

GENERALIZED BLOCH SPACES, INTEGRAL MEANS AND DIRICHLET-TYPE ENERGY INTEGRALS OF HYPERBOLIC HARMONIC MAPPINGS IN THE UNIT BALL

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ABSTRACT. In this paper, we investigate the properties of hyperbolic harmonic mappings in the unit ball \mathbb{B}^n in \mathbb{R}^n ($n \geq 2$). Firstly, we establish necessary and sufficient conditions for a hyperbolic harmonic mapping to be in the Bloch space $\mathcal{B}(\mathbb{B}^n)$ and the generalized Bloch space $\mathcal{L}_{\infty, \omega} \mathcal{B}_{\alpha, a}^0(\mathbb{B}^n)$, respectively. Secondly, we discuss the relationship between the integral means of hyperbolic harmonic mappings and that of their gradients. The obtained results are the generalizations of Hardy and Littlewood's related ones in the setting of hyperbolic harmonic mappings. Thirdly, we study the Dirichlet-type energy integrals of hyperbolic harmonic mappings. Finally, we characterize the weak uniform boundedness property of hyperbolic harmonic mappings in terms of the quasihyperbolic metric.

1. INTRODUCTION AND MAIN RESULTS

For $n \geq 2$, let $\mathbb{B}^n(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$, $\mathbb{S}^{n-1}(x_0, r) = \partial \mathbb{B}^n(x_0, r)$ and $\overline{\mathbb{B}}^n(x_0, r) = \mathbb{B}^n(x_0, r) \cup \mathbb{S}^{n-1}(x_0, r)$. In particular, we write $\mathbb{B}^n = \mathbb{B}^n(0, 1)$, $\mathbb{S}^{n-1} = \mathbb{S}^{n-1}(0, 1)$ and $\overline{\mathbb{B}}^n = \overline{\mathbb{B}}^n \cup \mathbb{S}^{n-1}$.

The purpose of this paper is to consider the hyperbolic harmonic mappings whose definition is as follows.

Definition 1.1. A mapping $u = (u_1, \dots, u_n) \in C^2(\mathbb{B}^n, \mathbb{R}^n)$ is said to be *hyperbolic harmonic* if

$$\Delta_h u = (\Delta_h u_1, \dots, \Delta_h u_n) = 0,$$

that is, for each $j \in \{1, \dots, n\}$, u_j satisfies the hyperbolic Laplace equation

$$\Delta_h u_j = 0,$$

where

$$(1.1) \quad \Delta_h u_j(x) = (1 - |x|^2)^2 \Delta u_j(x) + 2(n-2)(1 - |x|^2) \sum_{i=1}^n x_i \frac{\partial u_j}{\partial x_i}(x).$$

We refer to [4, 28, 40, 41] for basic properties of this class of mappings. For convenience, in the following of this paper, we always use the notation $\Delta_h u = 0$ to mean that $u = (u_1, \dots, u_n)$ is hyperbolic harmonic in \mathbb{B}^n .

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Obviously, for $n = 2$, hyperbolic harmonic mappings coincide with harmonic mappings. See [9, 11] and the references therein for the basic properties of harmonic mappings.

1.1. Generalized Hardy spaces and generalized Bloch spaces.

Definition 1.2. For $p \in (0, \infty]$, the *generalized Hardy space* $\mathcal{H}_g^p(\mathbb{B}^n)$ consists of all those functions $f : \mathbb{B}^n \rightarrow \mathbb{R}^n$ such that f is measurable, $M_p(r, f)$ exists for all $r \in (0, 1)$ and

$$\|f\|_p < \infty,$$

where

$$\|f\|_p = \sup_{0 < r < 1} \{M_p(r, f)\}$$

and

$$M_p(r, f) = \begin{cases} \left(\int_{\mathbb{S}^{n-1}} |f(r\xi)|^p d\sigma(\xi) \right)^{\frac{1}{p}}, & \text{if } p \in (0, \infty), \\ \sup_{\xi \in \mathbb{S}^{n-1}} \{|f(r\xi)|\}, & \text{if } p = \infty. \end{cases}$$

Here and hereafter, $d\sigma$ always denotes the normalized surface measure on \mathbb{S}^{n-1} so that $\sigma(\mathbb{S}^{n-1}) = 1$.

The classical *Hardy space* $\mathcal{H}^p(\mathbb{D})$ consisting of related analytic functions is a subspace of $\mathcal{H}_g^p(\mathbb{D})$, where \mathbb{D} denotes the unit disk in the complex plane \mathbb{C} (In this paper, we always identify \mathbb{R}^2 with \mathbb{C} and \mathbb{B}^2 with \mathbb{D} , respectively).

In order to introduce the definition of the generalized Bloch space, we need the following notion.

A continuous increasing function $\omega : [0, \infty) \rightarrow [0, \infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$ (cf. [12, 13, 24, 25]).

Given a subset Ω of \mathbb{R}^n , a function $f : \Omega \rightarrow \mathbb{R}^n$ is said to belong to the *Lipschitz space* $\mathcal{L}_\omega(\Omega)$ if there is a positive constant μ_{01} such that for all $x, y \in \Omega$,

$$|f(x) - f(y)| \leq \mu_{01}\omega(|x - y|).$$

First, we define the Bloch space of \mathbb{B}^n , denoted by $\mathcal{B}(\mathbb{B}^n)$, as the space of functions f in $C^1(\mathbb{B}^n, \mathbb{R}^n)$ such that

$$\|f\|^{\mathcal{B}} < \infty,$$

where

$$\|f\|^{\mathcal{B}} = \sup_{x \in \mathbb{B}^n} \{|Df(x)|(1 - |x|^2)\}$$

and $\|Df(x)\|$ denotes the matrix norm of the usual Jacobian matrix $Df(x)$ of f at x (See §2.1 below for the precise definition of $\|Df(x)\|$). This is only a semi-norm. Obviously, $\|f\|^{\mathcal{B}} = 0$ if and only if f is constant. The Bloch space $\mathcal{B}(\mathbb{B}^n)$ becomes a Banach space with the following norm:

$$\|f\|_{\mathcal{B}} = |f(0)| + \|f\|^{\mathcal{B}}.$$

For an analytic function f in \mathbb{D} , obviously, $\|Df(z)\| = |f'(z)|$. Therefore, the classical *Bloch space* $\mathcal{B}(\mathbb{D})$ consisting of the analytic functions f satisfying

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z \in \mathbb{D}} \{|f'(z)|(1 - |z|^2)\} < \infty$$

is a subspace of $\mathcal{B}(\mathbb{B}^n)$ (cf. [2, 18, 47]).

Next, we introduce the notion: *Generalized Bloch spaces*.

Definition 1.3. For $p \in (0, \infty]$, $\alpha > 0$, $\beta \in \mathbb{R}$ and a majorant ω , we use $\mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,a}^\beta(\mathbb{B}^n)$ to denote the *generalized Bloch space*, which consists of all functions $f \in C^1(\mathbb{B}^n, \mathbb{R}^n)$ with

$$\|f\|_{\mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,a}^\beta(\mathbb{B}^n)} < \infty,$$

where

$$\|f\|_{\mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,a}^\beta(\mathbb{B}^n)} = |f(0)| + \sup_{x \in \mathbb{B}^n} \left\{ M_p(|x|, \|Df\|) \omega(\phi_{\alpha,\beta,a}(x)) \right\},$$

$$\phi_{\alpha,\beta,a}(x) = (1 - |x|)^\alpha \left(\log \frac{a}{1 - |x|} \right)^\beta$$

and a is a constant satisfying

- (1) $a > 1$, if $\beta \leq 0$; and
- (2) $a \geq e^{\frac{\beta}{\alpha}}$, if $\beta > 0$.

The space $\mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,e}^\beta(\mathbb{D})$ was discussed in [5] and the space $\mathcal{L}_{p,id}\mathcal{B}_{\alpha,e^{\beta/\alpha}}^\beta(\mathbb{D})$ of analytic functions was introduced in [36]. Note that, when $\beta = 0$, the space $\mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,a}^0(\mathbb{B}^n)$ has nothing to do with the parameter a .

Obviously, $\mathcal{L}_{\infty,id}\mathcal{B}_{1,a}^0(\mathbb{B}^n) = \mathcal{B}(\mathbb{B}^n)$, where id denotes the identity mapping. Further, we have the following.

- (1) The special case $\mathcal{L}_{\infty,\omega}\mathcal{B}_{\alpha,a}^0(\mathbb{D})$ is called the ω - α -Bloch space (cf. [14, 46] and the related references therein).
- (2) The special case $\mathcal{L}_{\infty,\omega}\mathcal{B}_{1,a}^\beta(\mathbb{D})$ is called the *logarithmic ω -Bloch space* (cf. [35, 36] and the related references therein).
- (3) The special case $\mathcal{L}_{\infty,id}\mathcal{B}_{\alpha,a}^0(\mathbb{D})$ is called the *generalized α -Bloch space* (cf. [20, 21, 22] and the related references therein).
- (4) The special case $\mathcal{L}_{\infty,id}\mathcal{B}_{1,a}^\beta(\mathbb{D})$ is called the *generalized logarithmic Bloch space* (cf. [27, 46] and the related references therein).

In [12], Dyakonov discussed the relationship between the Lipschitz space $\mathcal{L}_\omega(\mathbb{D})$ and the bounded mean oscillation on analytic functions in \mathbb{D} ([12, Theorem 1]). In [5] and [7], the authors extended [12, Theorem 1] to the case of complex-valued harmonic mappings ([5, Theorem 4] and [7, Theorem 3]). Recently, Chen and Rasila generalized [5, Theorem 4], [7, Theorem 3] and [12, Theorem 1] to the setting of the solutions to the non-homogenous Yukawa PDE

$$(1.2) \quad \Delta f = \lambda f,$$

where $\lambda : \mathbb{B}^n \rightarrow \mathbb{R}$ is a nonnegative continuous function with $\sup_{x \in \mathbb{B}^n} \{\lambda(x)\} < \infty$ ([6, Theorem 1]). As the first aim of this paper, we consider the similar results of the above type for hyperbolic harmonic mappings. The following is our first result in this line.

Theorem 1.1. Suppose $\alpha \in [1, 2)$, ω is a majorant and $\Delta_h u = 0$. Then $u \in \mathcal{L}_{\infty,\omega}\mathcal{B}_{\alpha,a}^0(\mathbb{B}^n)$ if and only if there is a positive constant μ_1 such that for all $r \in$

$(0, 1 - |x|]$,

$$\frac{1}{|\mathbb{B}^n(x, r)|} \int_{\mathbb{B}^n(x, r)} |u(y) - u(x)| d\nu(y) \leq \frac{\mu_1}{\omega(r^\alpha)} r,$$

where $|\mathbb{B}^n(x, r)|$ means the Lebesgue volume of the ball $\mathbb{B}^n(x, r)$ and $d\nu$ denotes the normalized Lebesgue volume measure in \mathbb{B}^n .

Remark 1.1. If we take $n = 2$, $\alpha = 1$ and replace $\omega(t)$ by $\frac{1}{\omega(\frac{1}{t})}$ in Theorem 1.1, then Theorem 1.1 coincides with [7, Theorem 3].

To state our next result, let us recall the following notion.

The *hyperbolic distance* between two points x and y in \mathbb{B}^n is defined by

$$\rho(x, y) = \inf_{\gamma \in \Gamma_{xy}(\mathbb{B}^n)} \int_{\gamma} \frac{2}{1 - |z|^2} ds(z),$$

where ds is the length element on γ and $\Gamma_{xy}(\mathbb{B}^n)$ stands for the collection of all rectifiable curves in \mathbb{B}^n joining x and y (cf. [42]). See §2.2 below for more properties of ρ .

In [47], Zhu characterized the holomorphic Bloch space in \mathbb{C}^n in terms of the Bergman metric ([47, Theorem 3.6 and Corollary 3.7]). Now, we establish the following necessary and sufficient condition for hyperbolic harmonic mappings to be in $\mathcal{B}(\mathbb{B}^n)$ in terms of the hyperbolic metric.

Theorem 1.2. *Suppose $\Delta_h u = 0$. Then $u \in \mathcal{B}(\mathbb{B}^n)$ if and only if there exists a positive constant μ_2 such that for all $x, y \in \mathbb{B}^n$,*

$$(1.3) \quad |u(x) - u(y)| \leq \mu_2 \rho(x, y).$$

1.2. Integral means. First, we recall the following well known result on analytic functions due to Hardy and Littlewood.

Theorem A. ([15, Theorems 2 and 3] and [16, Theorem 46] or [10, Theorem 5.5]) *Suppose $p \in (0, \infty]$, $\alpha \in (1, \infty)$ and f is an analytic function in \mathbb{D} . Then the following statements are equivalent.*

- (1) $M_p(r, f') = O\left(\frac{1}{(1-r)^\alpha}\right)$ as $r \rightarrow 1$;
- (2) $M_p(r, f) = O\left(\frac{1}{(1-r)^{\alpha-1}}\right)$ as $r \rightarrow 1$.

Obviously, the above result of Hardy and Littlewood provides a close relationship between the integral means of analytic functions and that of their derivatives.

As the second aim of this paper, we consider Theorem A in the setting of hyperbolic harmonic mappings. Our first result is the following analog of the implication from (1) to (2) in Theorem A for hyperbolic harmonic mappings.

Theorem 1.3. *Suppose $\Delta_h u = 0$ and $u \in \mathcal{L}_{p, \omega} \mathcal{B}_{\alpha, a}^\beta(\mathbb{B}^n)$. Then for $r \in [0, 1)$ and $p \in [1, \infty]$,*

$$M_p(r, u) \leq |u(0)| + \frac{(\log a)^\beta \|u\|_{\mathcal{L}_{p, \omega} \mathcal{B}_{\alpha, a}^\beta(\mathbb{B}^n)}}{\omega((\log a)^\beta)} \int_0^1 \frac{r}{\phi_{\alpha, \beta, a}(rt)} dt.$$

Remark 1.2. By taking $n = 2$, $\omega(t) = t$, $\alpha > 1$ and $\beta = 0$, we obtain that Theorem 1.3 is a generalization of the implication from (1) to (2) in Theorem A even in the case of harmonic mappings when $p \in [1, \infty]$.

We also consider the converse of Theorem 1.3, and we get the following analog of the implication from (2) to (1) in Theorem A for hyperbolic harmonic mappings.

Theorem 1.4. Suppose $\Delta_h u = 0$ and

$$M_p(r, u) = O\left(\frac{1}{(1-r)^\alpha}\right) \text{ as } r \rightarrow 1,$$

where $p \in (0, \infty)$ and $\alpha > \frac{1}{p}$. Then for $q \in (0, \infty]$,

$$u \in \mathcal{L}_{q, id} \mathcal{B}_{\alpha+1+\frac{n-1}{p}, a}^0(\mathbb{B}^n).$$

1.3. Dirichlet-type energy integrals. For $\mu, \gamma, t \in \mathbb{R}$ and $f \in C^1(\mathbb{B}^n, \mathbb{R}^n)$, the integral

$$\mathcal{D}_f(\mu, \gamma, t) = \int_{\mathbb{B}^n} (1 - |x|)^\mu |f(x)|^\gamma \cdot \|Df(x)\|^t d\nu(x) < \infty$$

is called the *Dirichlet-type energy integral* of f .

About the relationship between Dirichlet-type energy integrals and function spaces, the following are known.

- (1) For $f \in C^1(\mathbb{B}^n, \mathbb{R})$, the condition $\mathcal{D}_f(\mu, 0, 2) < \infty$ implies that f belongs to the Dirichlet space (cf. [41, Section 6]).
- (2) For a holomorphic function $f : \mathbb{B}^{2n} \rightarrow \mathbb{C}$, the condition $\mathcal{D}_f(\mu, \gamma, 0) < \infty$ implies that f belongs to the weighted Bergman space, where \mathbb{B}^{2n} denotes the unit ball in \mathbb{C}^n (cf. [47, Page 39]).
- (3) In [44], Yamashita characterized the analytic Hardy space consisting of analytic functions from \mathbb{D} to \mathbb{C} in terms of the Dirichlet-type energy integrals ([44, Theorem 1]). Later, in [38], Stoll characterized the Hardy space consisting of holomorphic mappings from \mathbb{B}^{2n} to \mathbb{C} , also, in terms of the Dirichlet-type energy integrals ([38, Theorem 1]). In addition, the Dirichlet-type energy integrals were used to characterize the hyperbolic harmonic Hardy space consisting of mappings in $C^2(\mathbb{B}^{2n}, \mathbb{C})$ ([39, Theorem 6.18]).

Recently, in [7], the authors discussed the Dirichlet-type energy integrals of solutions of non-homogeneous Yukawa PDE. They proved that if $f \in C^2(\mathbb{B}^{2n}, \mathbb{C})$ is the solutions of (1.2), where λ is a nonnegative constant function, then there exist two positive constants μ_{02} and μ_{03} such that

$$\int_{\mathbb{B}^n} (1 - |z|)^{1+\frac{2}{\beta}(n-1)} \Delta(|f(z)|^{\frac{2}{\beta}}) d\nu(x) \leq \mu_{02} \mathcal{D}_f(\beta - 1, 1, 1) + \mu_{03},$$

where $\beta \in (0, 1]$ ([7, Theorem 4]).

For more properties and applications of the Dirichlet-type energy integrals, we refer to [5, 6, 7, 34, 41, 43, 44, 45] and the related references therein.

As the third aim of this paper, we consider Dirichlet-type energy integrals of hyperbolic harmonic mappings. Our result is as follows.

Theorem 1.5. Suppose $u \in C^2(\mathbb{B}^n, \mathbb{R}^n)$, $\Delta_h u = 0$, and

$$\lim_{r \rightarrow 1^-} (1-r)^\beta \int_{\mathbb{S}^{n-1}} |u(r\xi)| \cdot \|Du(r\xi)\| d\sigma(\xi) = 0,$$

where β is a constant satisfying

- (1) $\beta \geq 1$, if $n = 2$; and
- (2) $\beta > 2 + 2^{n-2}n - 2^{n-1} - n$, if $n \geq 3$.

Then there exists a positive constant μ_3 such that

$$\begin{aligned} & \int_{\mathbb{B}^n} (1-|x|)^{\beta n + \beta^2 - 2\beta - n} \Delta_h(|u(x)|^{2\beta}) d\nu(x) \\ & \leq \mu_3 \max \left\{ |u(0)|^{2\beta-2} \mathcal{D}_u(\beta-1, 1, 1), (\mathcal{D}_u(\beta-1, 1, 1))^\beta \right\}. \end{aligned}$$

1.4. Weak uniform boundedness property. Let Ω be a proper domain of \mathbb{R}^n . For $x \in \Omega$, we use $d_\Omega(x)$ to denote the Euclidean distance from x to the boundary $\partial\Omega$ of Ω . For $x, y \in \Omega$, let

$$r_\Omega(x, y) = \frac{|x-y|}{\min\{d_\Omega(x), d_\Omega(y)\}} \quad \text{and} \quad k_\Omega(x, y) = \inf_{\gamma \in \Gamma_{xy}(\Omega)} \int_\gamma \frac{1}{d_\Omega(z)} ds(z)$$

(cf. [23, 42]).

We say that $f : \Omega \rightarrow f(\Omega) \subset \mathbb{R}^n$ satisfies the *weak uniform boundedness property* in Ω (with respect to r_Ω) if there is a constant $\mu_{04} > 0$ such that for all $x, y \in \Omega$,

$$(1.4) \quad r_\Omega(x, y) \leq \frac{1}{2} \quad \text{implies} \quad r_{f(\Omega)}(f(x), f(y)) \leq \mu_{04}.$$

Remark 1.3. The above definition of the weak uniform boundedness property is equivalent to the following one: $f : \Omega \rightarrow f(\Omega)$ is said to satisfy the *weak uniform boundedness property* in Ω (with respect to r_Ω) if for any $\mu_{05} \in (0, 1)$, there is a constant $\mu_{06} > 0$ such that for all $x, y \in \Omega$,

$$r_\Omega(x, y) \leq \mu_{05} \quad \text{implies} \quad r_{f(\Omega)}(f(x), f(y)) \leq \mu_{06}.$$

In [23, Theorem 2.8], Mateljević and Vuorinen proved that a harmonic mapping f satisfies the weak uniform boundedness property in $G \subset \mathbb{R}^n$ if and only if there exists a constant μ_{07} such that for all $x, y \in G$,

$$k_{f(G)}(f(x), f(y)) \leq \mu_{07} k_G(x, y).$$

Recently, Chen and Rasila generalized this result to the case of the solutions to Yukawa equation in \mathbb{B}^n ([6, Theorem 2]). As the last aim of this paper, we consider the weak uniform boundedness property of hyperbolic harmonic mappings. Our result reads as follows.

Theorem 1.6. Suppose $\Delta_h u = 0$. Then u satisfies the weak uniform boundedness property in \mathbb{B}^n if and only if there exists a positive constant μ_4 such that for all $x, y \in \mathbb{B}^n$,

$$k_{u(\mathbb{B}^n)}(u(x), u(y)) \leq \mu_4 k_{\mathbb{B}^n}(x, y).$$

This paper is organized as follows. In Section 2, some necessary terminology and notation will be introduced. In Section 3, we shall prove Theorem 1.1, and in Section 4, we shall show Theorem 1.2. The proofs of Theorems 1.3 and 1.4 will be presented in Section 5. Section 6 will be devoted to the proof of Theorem 1.5, and we shall demonstrate Theorem 1.6 in the last section, Section 7.

2. PRELIMINARIES

In this section, we recall some necessary terminology and notation.

2.1. Matrix notations. For a natural number n , let

$$A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}.$$

For $A \in \mathbb{R}^{n \times n}$, denote by $\|A\|$ the matrix norm

$$\|A\| = \sup\{|Ax| : x \in \mathbb{R}^n, |x| = 1\},$$

and by $l(A)$ the matrix function

$$l(A) = \inf\{|Ax| : x \in \mathbb{R}^n, |x| = 1\}.$$

For a domain $\Omega \subset \mathbb{R}^n$, let $f = (f_1, \dots, f_n) : \Omega \rightarrow \mathbb{R}^n$ be a function that has all partial derivatives at $x = (x_1, \dots, x_n)$ in Ω . Then $Df(x)$ denotes the usual Jacobian matrix

$$Df = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x)}{\partial x_1} & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{pmatrix} = (\nabla f_1(x) \cdots \nabla f_n(x))^T$$

at x , where T is the transpose and the gradients $\nabla f_j(x)$ are understood as column vectors (cf. [4]). For two column vectors $x, y \in \mathbb{R}^n$, we use $\langle x, y \rangle$ to denote the inner product of x and y . For $j \in \{1, \dots, n\}$, it follows from

$$\|Df(x)\|^2 = \sup_{\xi \in \mathbb{S}^{n-1}} \left\{ \sum_{j=1}^n (\langle \nabla f_j(x), \xi \rangle)^2 \right\}$$

that

$$(2.1) \quad \|Df(x)\|^2 \leq \sum_{j=1}^n |\nabla f_j(x)|^2 = \sum_{i=1}^n \left| \frac{\partial}{\partial x_i} f(x) \right|^2 \quad \text{and} \quad \|Df(x)\| \geq |\nabla f_j(x)|.$$

2.2. Hyperbolic metric. For any $w \in \mathbb{B}^n$, let

$$\varphi_w(x) = \frac{|x - w|^2 w - (1 - |w|^2)(x - w)}{[x, w]^2}$$

in \mathbb{B}^n , where $[x, w] = \left| |x|w - \frac{x}{|x|} \right|$. Then φ_w is a Möbius transformation from \mathbb{B}^n onto \mathbb{B}^n with $\varphi_w(w) = 0$, $\varphi_w(0) = w$ and $\varphi_w(\varphi_w(x)) = x$.

We denote by $\mathcal{M}(\mathbb{B}^n)$ the set of all Möbius transformations in \mathbb{B}^n . It is well known that if $\varphi \in \mathcal{M}(\mathbb{B}^n)$, then there exist $w \in \mathbb{B}^n$ and an orthogonal transformation A such that

$$\varphi(x) = A\varphi_w(x)$$

(cf. [40, Theorem 2.1]).

For more information on Möbius transformations in \mathbb{B}^n , see e.g. [1, 3, 42].

In terms of φ_w , the *hyperbolic metric* ρ in \mathbb{B}^n can be given by

$$(2.2) \quad \rho(x, w) = \log \frac{1 + |\varphi_w(x)|}{1 - |\varphi_w(x)|}$$

for $x, w \in \mathbb{B}^n$. By [40, Equation (2.6)] or [4, Equation (2.11)], we see that for $x, w \in \mathbb{B}^n$,

$$(2.3) \quad |\varphi_w(x)| = |\varphi_x(w)| = \frac{|x - w|}{[x, w]}.$$

Therefore, $\rho(x, w) = \rho(w, x)$. In particular,

$$\rho(0, x) = \log \frac{1 + |x|}{1 - |x|} \quad \text{and} \quad \rho(x, y) = \rho(\varphi(x), \varphi(y))$$

for all $\varphi \in \mathcal{M}(\mathbb{B}^n)$ (cf. [42, Chapter 1]).

For any $w \in \mathbb{B}^n$ and $0 < r < 1$, we define the *pseudo-hyperbolic ball* with center w and radius r as

$$(2.4) \quad E(w, r) = \{x \in \mathbb{B}^n : |\varphi_w(x)| < r\}.$$

Clearly, $E(w, r) = \varphi_w(\mathbb{B}^n(0, r))$ (cf. [41]).

It is well known that $E(w, r)$ is also a Euclidean ball with centre c_w and radius r_w which are as follows:

$$(2.5) \quad c_w = \frac{1 - r^2}{1 - |w|^2 r^2} w \quad \text{and} \quad r_w = \frac{1 - |w|^2}{1 - |w|^2 r^2} r.$$

By [29, Lemma 2.1], we obtain that for any $\delta \in (0, 1)$ and $y \in E(x, \delta)$,

$$1 - |y|^2 \leq \left| |x|y - \frac{x}{|x|} \right| (1 + |y|) \leq \frac{2(1 + \delta)}{1 - \delta} (1 - |x|^2).$$

By (2.3) and (2.4), we see that $y \in E(x, \delta)$ if and only if $x \in E(y, \delta)$. Therefore,

$$(2.6) \quad 1 - |x|^2 \leq \frac{2(1 + \delta)}{1 - \delta} (1 - |y|^2).$$

2.3. Hyperbolic harmonic mappings. For all $\varphi \in \mathcal{M}(\mathbb{B}^n)$ and $f \in C^2(\mathbb{B}^n, \mathbb{R}^n)$, we have the following Möbius invariance property (cf. [41, Section 2]):

$$(2.7) \quad \Delta_h(f \circ \varphi)(x) = \Delta_h f(\varphi(x)).$$

Obviously, (1.1) implies that

$$\Delta_h f(x) = \Delta_h f(\varphi_x(0)) = \Delta_h(f \circ \varphi_x)(0) = \Delta(f \circ \varphi_x)(0).$$

It is well known that if $\psi \in C(\mathbb{S}^{n-1}, \mathbb{R}^n)$, then the Dirichlet problem

$$\begin{cases} \Delta_h f = 0 & \text{in } \mathbb{B}^n, \\ f = \psi & \text{on } \mathbb{S}^{n-1} \end{cases}$$

has a unique solution in $C(\overline{\mathbb{B}^n})$ and can be represented by

$$f(x) = P_h[\psi](x) = \int_{\mathbb{S}^{n-1}} P_h(x, \xi) \psi(\xi) d\sigma(\xi)$$

(cf. [8] or [40]), where

$$P_h(x, \xi) = \left(\frac{1 - |x|^2}{|x - \xi|^2} \right)^{n-1}.$$

For $f \in C^1(\mathbb{B}^n, \mathbb{R})$, the gradient $\nabla^h f$ with respect to the hyperbolic metric is given by

$$\nabla^h f(x) = (1 - |x|^2) \nabla f(x)$$

(cf. [26, 28, 40]). In particular,

$$(2.8) \quad |\nabla^h f(x)| = (1 - |x|^2) |\nabla f(x)|.$$

Furthermore, for all $\varphi \in \mathcal{M}(\mathbb{B}^n)$, by [40, Theorem 3.2] or [41, Equation (2.14)], we have

$$(2.9) \quad |\nabla^h(f \circ \varphi)(x)| = |\nabla^h f(\varphi(x))|.$$

From [41, Lemma 3.3] or [26, Theorem 2.1], we can easily obtain the following useful result.

Lemma B. *Suppose $p \in (0, \infty)$ and $\delta \in (0, \frac{1}{2})$. Then there exists a constant μ_{08} such that for any $f \in C^2(\mathbb{B}^n, \mathbb{R}^n)$ with $\Delta_h f = 0$,*

- (1) $|\nabla^h f(x)|^p \leq \mu_{08} \delta^{-n} \int_{E(x, \delta)} |\nabla^h f(y)|^p d\tau(y)$ in \mathbb{B}^n , and
- (2) $|\nabla^h f(x)|^p \leq \mu_{08} \delta^{-n} \int_{E(x, \delta)} |f(y)|^p d\tau(y)$ in \mathbb{B}^n ,

where $\mu_{08} = \mu_{08}(p, \delta)$ (which means that the constant μ_{08} depends only on the given parameters p and δ) and $d\tau$ denotes the Möbius invariant measure in \mathbb{B}^n , which is given by

$$d\tau(x) = \frac{d\nu(x)}{(1 - |x|^2)^n}.$$

3. GENERALIZED BLOCH SPACES AND BOUNDED MEAN OSCILLATION

The purpose of this section is to prove Theorem 1.1. Before the proof, we need the following lemma.

Lemma 3.1. *Suppose $\Delta_h u = 0$. Then there is a constant μ_{09} such that for any $x \in \mathbb{B}^n$,*

$$||Du(x)|| \leq \frac{5^n \sqrt{n} \mu_{09}}{2^n (1 - |x|)^{n+1}} \int_{\mathbb{B}^n(x, \frac{1}{4}(1 - |x|))} |u(y) - u(x)| d\nu(y),$$

where μ_{09} is a constant depending only on $\mu_{08}(1, \frac{1}{9})$ and μ_{08} is the constant from Lemma B.

Proof. For $x \in \mathbb{B}^n$, by (2.1), we know that

$$(3.1) \quad \|Du(x)\| \leq \left(\sum_{j=1}^n |\nabla u_j(x)|^2 \right)^{\frac{1}{2}}.$$

Hence, to prove this lemma, it suffices to estimate the quantity $|\nabla u_j(x)|$. Since for a fixed $x \in \mathbb{B}^n$,

$$\Delta_h(u(w) - u(x)) = 0$$

in \mathbb{B}^n , by taking $p = 1$ and $\delta = \frac{1}{9}$, we see from Lemma B(2) and (2.8) that there is a constant μ_{10} such that for any $w \in \mathbb{B}^n$ and each $j \in \{1, \dots, n\}$,

$$(3.2) \quad \begin{aligned} (1 - |w|^2) |\nabla u_j(w)| &\leq \mu_{10} \int_{E(w, \frac{1}{9})} |u_j(y) - u_j(x)| d\tau(y) \\ &\leq \mu_{10} \int_{E(w, \frac{1}{9})} |u(y) - u(x)| d\tau(y). \end{aligned}$$

Therefore, (3.1) guarantees the following:

$$(3.3) \quad \begin{aligned} (1 - |w|^2) \|Du(w)\| &\leq (1 - |w|^2) \left(\sum_{j=1}^n |\nabla u_j(w)|^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{n} \mu_{10} \int_{E(w, \frac{1}{9})} |u(y) - u(x)| d\tau(y) \\ &= \sqrt{n} \mu_{10} \int_{E(w, \frac{1}{9})} \frac{|u(y) - u(x)|}{(1 - |y|^2)^n} d\nu(y). \end{aligned}$$

Moreover, by (2.5), we know that

$$(3.4) \quad E(x, \frac{1}{9}) = \mathbb{B}^n \left(\frac{80}{81 - |x|^2} x, \frac{9(1 - |x|^2)}{81 - |x|^2} \right) \subset \mathbb{B}^n \left(x, \frac{1}{4}(1 - |x|) \right).$$

Further, for any $y \in E(x, \frac{1}{9})$, (2.6) implies

$$(3.5) \quad \frac{1}{1 - |y|^2} \leq \frac{5}{2(1 - |x|)}.$$

By letting $w = x$, it follows from (3.3)~(3.5) that

$$(1 - |x|^2) \|Du(x)\| \leq \frac{5^n \sqrt{n} \mu_{10}}{2^n (1 - |x|)^n} \int_{\mathbb{B}^n(x, \frac{1}{4}(1 - |x|))} |u(y) - u(x)| d\nu(y).$$

By taking $\mu_{09} = \mu_{10}$, we know that the lemma is proved. \square

Proof of Theorem 1.1. First, we show the “if” part in the theorem. For any $x \in \mathbb{B}^n$, we see that

$$\frac{1}{2^n(1-|x|)^{n+1}} = \frac{1}{8^n(1-|x|)\left(\frac{1}{4}(1-|x|)\right)^n} = \frac{|\mathbb{B}^n|}{8^n(1-|x|) \left|\mathbb{B}^n\left(x, \frac{1}{4}(1-|x|)\right)\right|}.$$

Then by Lemma 3.1, we know that there is a constant μ_{09} such that for any $x \in \mathbb{B}^n$,

$$\|Du(x)\| \leq \frac{5^n \sqrt{n} \mu_{09} |\mathbb{B}^n|}{8^n(1-|x|) \left|\mathbb{B}^n\left(x, \frac{1}{4}(1-|x|)\right)\right|} \int_{\mathbb{B}^n\left(x, \frac{1}{4}(1-|x|)\right)} |u(y) - u(x)| d\nu(y).$$

Then the assumption in the theorem implies

$$\|Du(x)\| \leq \frac{5^n \sqrt{n} \mu_1 \mu_{09}}{2^{3n+2} \omega\left(\frac{(1-|x|)^\alpha}{4^\alpha}\right)} |\mathbb{B}^n|.$$

Moreover, it follows from $\alpha \in [1, 2)$ and [8, Lemma 2.2(2)] that

$$\omega\left(4^\alpha \cdot \frac{(1-|x|)^\alpha}{4^\alpha}\right) \leq 4^\alpha \omega\left(\frac{(1-|x|)^\alpha}{4^\alpha}\right),$$

and thus, for all $x \in \mathbb{B}^n$,

$$\|Du(x)\| \leq \frac{5^n \sqrt{n} \mu_1 \mu_{09}}{2^{3n+2-2\alpha} \omega\left((1-|x|)^\alpha\right)} |\mathbb{B}^n|,$$

which means that $u \in \mathcal{L}_{\infty, \omega} \mathcal{B}_{\alpha, a}^0(\mathbb{B}^n)$.

Next, we prove the “only if” part. Let $x \in \mathbb{B}^n$. For any $r \in (0, 1 - |x|]$ and $y \in \mathbb{B}^n(x, r)$, obviously, we have

$$(3.6) \quad |y - x| < r \leq 1 - |x| \quad \text{and} \quad t|x - y| < 1 - |x|,$$

where $t \in [0, 1]$. For the proof, we need an upper bound on the quantity $|u(y) - u(x)|$. For this, we let $\gamma_{[x, y]}$ denote the segment between x and y with the parametrization $\gamma(t) = (1 - t)x + ty$, where $t \in [0, 1]$. By the well-known gradient theorem (see, e.g. [31, Theorem 6.24]), we have that for each $j \in \{1, \dots, n\}$,

$$\int_{\gamma_{[x, y]}} \langle \nabla u_j(\gamma), d\gamma \rangle = \int_0^1 \langle \nabla u_j(\gamma(t)), \gamma'(t) \rangle dt = \int_0^1 \frac{d}{dt} (u_j \circ \gamma(t)) dt = u_j(y) - u_j(x).$$

Note that

$$Du(\gamma(t)) \times \gamma'(t) = \begin{pmatrix} \langle \nabla u_1(\gamma(t)), \gamma'(t) \rangle \\ \vdots \\ \langle \nabla u_n(\gamma(t)), \gamma'(t) \rangle \end{pmatrix},$$

where $A \times B$ denotes the product of two matrices A and B . Hence

$$u(y) - u(x) = \int_0^1 Du(\gamma(t)) \times \gamma'(t) dt,$$

and therefore

$$(3.7) \quad |u(y) - u(x)| = \left| \int_0^1 Du(\gamma(t)) \times \gamma'(t) dt \right| \leq \int_{\gamma_{[x, y]}} \|Du(\gamma)\| \cdot |d\gamma|.$$

Moreover, it follows from the assumption $u \in \mathcal{L}_{\infty, \omega} \mathcal{B}_{\alpha, a}^0(\mathbb{B}^n)$ that for $x \in \mathbb{B}^n$,

$$\|Du(x)\| \leq \frac{1}{\omega((1-|x|)^\alpha)} \|u\|_{\mathcal{L}_{\infty, \omega} \mathcal{B}_{\alpha, a}^0(\mathbb{B}^n)}.$$

We infer from the easy fact $|\gamma'(t)| = |x - y|$ that

$$\begin{aligned} |u(y) - u(x)| &\leq \|u\|_{\mathcal{L}_{\infty, \omega} \mathcal{B}_{\alpha, a}^0(\mathbb{B}^n)} \int_0^1 \frac{|x - y|}{\omega((1 - |(1-t)x + ty|)^\alpha)} dt \\ &\leq \|u\|_{\mathcal{L}_{\infty, \omega} \mathcal{B}_{\alpha, a}^0(\mathbb{B}^n)} \int_0^1 \frac{|x - y|}{\omega((1 - |x| - t|x - y|)^\alpha)} dt \\ &= \|u\|_{\mathcal{L}_{\infty, \omega} \mathcal{B}_{\alpha, a}^0(\mathbb{B}^n)} \int_0^{|x-y|} \frac{1}{\omega((1 - |x| - t)^\alpha)} dt, \end{aligned}$$

since the inequalities (3.6) and the assumption $\alpha \in [1, 2)$ guarantee that

$$\omega((1 - |(1-t)x + ty|)^\alpha) \geq \omega((1 - |x| - t|y - x|)^\alpha).$$

This is our needed upper bound on $|u(y) - u(x)|$.

Now, we are ready to finish the proof. Let $x - y = \eta \in \mathbb{B}^n$. A similar argument as in the proof of [6, Theorem 1] implies that

$$\begin{aligned} &\frac{1}{|\mathbb{B}^n(x, r)|} \int_{\mathbb{B}^n(x, r)} |u(y) - u(x)| d\nu(y) \\ &\leq \frac{\|u\|_{\mathcal{L}_{\infty, \omega} \mathcal{B}_{\alpha, a}^0(\mathbb{B}^n)}}{|\mathbb{B}^n(x, r)|} \int_{\mathbb{B}^n(x, r)} \left(\int_0^{|x-y|} \frac{1}{\omega((1 - |x| - t)^\alpha)} dt \right) d\nu(y) \\ &= \frac{\|u\|_{\mathcal{L}_{\infty, \omega} \mathcal{B}_{\alpha, a}^0(\mathbb{B}^n)}}{|\mathbb{B}^n(0, r)|} \int_{\mathbb{B}^n(0, r)} \left(\int_0^{|\eta|} \frac{1}{\omega((1 - |x| - t)^\alpha)} dt \right) d\nu(\eta) \\ &\leq \frac{n\|u\|_{\mathcal{L}_{\infty, \omega} \mathcal{B}_{\alpha, a}^0(\mathbb{B}^n)}}{(2 - \alpha)} \frac{r}{\omega(r^\alpha)}, \end{aligned}$$

which is what we need. \square

4. BLOCH SPACE AND HYPERBOLIC METRIC

The aim of this section is to prove Theorem 1.2. We start this section with a lemma.

Lemma 4.1. *Suppose $\Delta_h u = 0$ and $u \in \mathcal{B}(\mathbb{B}^n)$. Then for any $z \in \mathbb{B}^n$,*

$$|u(z) - u(0)| \leq \frac{\|u\|_{\mathcal{B}}}{2} \log \frac{1 + |z|}{1 - |z|}.$$

Proof. For any $z \in \mathbb{B}^n$, we let $\gamma_{[0, z]}$ denote the segment between 0 and z with the parametrization $\gamma(t) = tz$, where $t \in [0, 1]$. A similar argument as in (3.7) leads to

$$|u(z) - u(0)| \leq \int_{\gamma_{[0, z]}} \|Du(\gamma)\| \cdot |d\gamma| = |z| \int_0^1 \|Du(tz)\| dt.$$

Then the assumption

$$\|u\|^{\mathcal{B}} = \sup_{x \in \mathbb{B}^n} (1 - |x|^2) \|Du(x)\| < \infty$$

implies that

$$|u(z) - u(0)| \leq \|u\|^{\mathcal{B}} \int_0^1 \frac{|z|}{1 - |tz|^2} dt = \frac{\|u\|^{\mathcal{B}}}{2} \log \frac{1 + |z|}{1 - |z|},$$

as needed. \square

Proof of Theorem 1.2. First, we show the “if” part in the theorem. Since

$$\|u\|_{\mathcal{B}} = |u(0)| + \sup_{w \in \mathbb{B}^n} \{(1 - |w|^2) \|Du(w)\|\},$$

we know that, to prove this part, we need to estimate the quantity $(1 - |w|^2) \|Du(w)\|$. Since for a fixed $x \in \mathbb{B}^n$,

$$\Delta_h(u(w) - u(x)) = 0$$

in \mathbb{B}^n , by taking $p = 1$ and $\delta = \frac{1}{9}$, we know from Lemma B(2), (3.1) and (3.2) that there is a constant μ_{10} such that for any $w \in \mathbb{B}^n$ and each $j \in \{1, \dots, n\}$,

$$\begin{aligned} (1 - |w|^2)^2 \|Du(w)\|^2 &\leq (1 - |w|^2)^2 \sum_{j=1}^n |\nabla u_j(w)|^2 \\ &\leq \mu_{10}^2 \sum_{j=1}^n \left(\int_{E(w, \frac{1}{9})} |u_j(y) - u_j(x)| d\tau(y) \right)^2. \end{aligned}$$

Further, Hölder inequality leads to

$$\begin{aligned} &(1 - |w|^2)^2 \|Du(w)\|^2 \\ &\leq \mu_{10}^2 \left(\int_{E(w, \frac{1}{9})} d\tau(y) \right) \cdot \left(\int_{E(w, \frac{1}{9})} \sum_{j=1}^n |u_j(y) - u_j(x)|^2 d\tau(y) \right) \\ &= \mu_{10}^2 \tau(E(w, \frac{1}{9})) \int_{E(w, \frac{1}{9})} |u(y) - u(x)|^2 d\tau(y) \\ &\leq \mu_{10}^2 \left(\tau(E(w, \frac{1}{9})) \right)^2 \sup_{y \in E(w, \frac{1}{9})} \{|u(y) - u(x)|^2\}. \end{aligned}$$

Moreover, by [41, Equation (2.7)], we know

$$\tau\left(E(w, \frac{1}{9})\right) = n \int_0^{\frac{1}{9}} t^{n-1} (1 - t^2)^{-n} dt \leq \frac{9^n}{80^n}.$$

By letting $w = x$, we see that

$$\begin{aligned}
(1 - |x|^2)^2 \|Du(x)\|^2 &\leq \mu_2^2 \mu_{10}^2 \frac{81^n}{80^{2n}} \sup_{y \in E(x, \frac{1}{9})} \{\rho^2(y, x)\} && \text{(by (1.3))} \\
&= \mu_2^2 \mu_{10}^2 \frac{81^n}{80^{2n}} \sup_{y \in E(x, \frac{1}{9})} \left\{ \log^2 \left(\frac{1 + |\varphi_x(y)|}{1 - |\varphi_x(y)|} \right) \right\} && \text{(by (2.2))} \\
&\leq \mu_2^2 \mu_{10}^2 \frac{81^n}{80^{2n}} \log^2 \frac{10}{8}. && \text{(by (2.4))}
\end{aligned}$$

Then the arbitrariness of x in \mathbb{B}^n ensures the following:

$$\|u\|_{\mathcal{B}} = |u(0)| + \sup_{x \in \mathbb{B}^n} \{(1 - |x|^2) \|Du(x)\|\} \leq |u(0)| + \mu_2 \mu_{10} \frac{9^n}{80^n} \log \frac{10}{8} < \infty.$$

Next, we prove the “only if” part. We shall show this part by applying Lemma 4.1 to the mapping $u \circ \varphi_y$, where $y \in \mathbb{B}^n$. Hence we have to verify that $u \circ \varphi_y$ satisfies the conditions in Lemma 4.1. Obviously, the hyperbolic harmonicity of $u \circ \varphi_y$ easily follows from (2.7). It remains to check that $u \circ \varphi_y$ satisfies the second assumption in Lemma 4.1, which is stated in the following claim.

Claim 4.1. *For any $y \in \mathbb{B}^n$, $u \circ \varphi_y \in \mathcal{B}(\mathbb{B}^n)$.*

Since $u \circ \varphi_y \in \mathcal{B}(\mathbb{B}^n)$ if and only if

$$\|u \circ \varphi_y\|_{\mathcal{B}} = \sup_{x \in \mathbb{B}^n} \{ \|D(u \circ \varphi_y)(x)\| (1 - |x|^2) \} < \infty,$$

clearly, to check this claim, it needs to estimate the quantity

$$\|D(u \circ \varphi_y)(x)\| (1 - |x|^2).$$

To reach this goal, we first obtain from (2.8) and (2.9) that for each $j \in \{1, \dots, n\}$,

$$\begin{aligned}
(4.1) \quad &\sup_{x \in \mathbb{B}^n} \{ (1 - |x|^2) |\nabla(u_j \circ \varphi_y)(x)| \} = \sup_{x \in \mathbb{B}^n} \{ |\nabla^h(u_j \circ \varphi_y)(x)| \} \\
&= \sup_{x \in \mathbb{B}^n} \{ |\nabla^h u_j(\varphi_y(x))| \} = \sup_{w \in \mathbb{B}^n} \{ |\nabla^h u_j(w)| \} \\
&= \sup_{w \in \mathbb{B}^n} \{ (1 - |w|^2) |\nabla u_j(w)| \} < \infty,
\end{aligned}$$

where in the last inequality, the assumption $u \in \mathcal{B}(\mathbb{B}^n)$ and (2.1) are exploited. Then

$$\begin{aligned}
(\|u \circ \varphi_y\|_{\mathcal{B}})^2 &= \sup_{x \in \mathbb{B}^n} \{ (1 - |x|^2)^2 \|D(u \circ \varphi_y)(x)\|^2 \} \\
&\leq \sup_{x \in \mathbb{B}^n} \left\{ (1 - |x|^2)^2 \sum_{j=1}^n |\nabla(u_j \circ \varphi_y)(x)|^2 \right\} && \text{(by (2.1))} \\
&\leq \sum_{j=1}^n \left(\sup_{x \in \mathbb{B}^n} \{ (1 - |x|^2)^2 |\nabla(u_j \circ \varphi_y)(x)|^2 \} \right) \\
&= \sum_{j=1}^n \left(\sup_{w \in \mathbb{B}^n} \{ (1 - |w|^2)^2 |\nabla u_j(w)|^2 \} \right). && \text{(by (4.1))}
\end{aligned}$$

Again, by (2.1) and the assumption $u \in \mathcal{B}(\mathbb{B}^n)$, we have

$$(4.2) \quad (||u \circ \varphi_y||^{\mathcal{B}})^2 \leq \sum_{j=1}^n \left(\sup_{w \in \mathbb{B}^n} \{ (1 - |w|^2)^2 ||Du(w)||^2 \} \right) = n(||u||^{\mathcal{B}})^2 < \infty.$$

Hence the claim is proved.

Now, we have known that $u \circ \varphi_y$ satisfies the conditions in Lemma 4.1, and so we are ready to finish the proof of the theorem by applying Lemma 4.1. For $x, y \in \mathbb{B}^n$, let $x = \varphi_y(z)$. Obviously, $z = \varphi_y(x) \in \mathbb{B}^n$. Then it follows that

$$\begin{aligned} |u(x) - u(y)| &= |u \circ \varphi_y(z) - u \circ \varphi_y(0)| \\ &\leq \frac{||u \circ \varphi_y||^{\mathcal{B}}}{2} \log \frac{1 + |z|}{1 - |z|} && \text{(by Lemma 4.1)} \\ &\leq \frac{\sqrt{n}||u||^{\mathcal{B}}}{2} \log \frac{1 + |\varphi_y(x)|}{1 - |\varphi_y(x)|} && \text{(by (4.2))} \\ &\leq \frac{\sqrt{n}||u||^{\mathcal{B}}}{2} \rho(x, y), && \text{(by (2.2))} \end{aligned}$$

as required. \square

5. GENERALIZED BLOCH SPACES AND INTEGRAL MEANS

The aim of this section is to prove Theorems 1.3 and 1.4. Before the proofs, we need some preparation.

Lemma 5.1. *Suppose $u \in \mathcal{L}_{p,\omega} \mathcal{B}_{\alpha,a}^{\beta}(\mathbb{B}^n)$. Then for $p \in (0, \infty]$ and $r \in [0, 1)$,*

$$M_p(r, ||Du||) \leq \frac{(\log a)^{\beta} ||u||_{\mathcal{L}_{p,\omega} \mathcal{B}_{\alpha,a}^{\beta}(\mathbb{B}^n)}}{\omega((\log a)^{\beta}) \phi_{\alpha,\beta,a}(r)}.$$

Proof. By direct calculations, we see that

$$\phi'_{\alpha,\beta,a}(r) = (1 - r)^{\alpha-1} \left(\log \frac{a}{1-r} \right)^{\beta-1} \left(\beta - \alpha \log \frac{a}{1-r} \right).$$

It follows from the assumptions in the lemma that $\phi'_{\alpha,\beta,a}(r) \leq 0$. Hence, $\phi_{\alpha,\beta,a}(r)$ is non-increasing in $(0, 1)$. Then the assumption “ $\frac{\omega(t)}{t}$ being non-increasing in $(0, 1)$ ” implies that $\frac{\phi_{\alpha,\beta,a}(r)}{\omega(\phi_{\alpha,\beta,a}(r))}$ is also non-increasing in $(0, 1)$. Therefore,

$$M_p(r, ||Du||) \leq \frac{||u||_{\mathcal{L}_{p,\omega} \mathcal{B}_{\alpha,a}^{\beta}(\mathbb{B}^n)}}{\omega(\phi_{\alpha,\beta,a}(r))} \leq \frac{||u||_{\mathcal{L}_{p,\omega} \mathcal{B}_{\alpha,a}^{\beta}(\mathbb{B}^n)} \phi_{\alpha,\beta,a}(0)}{\omega(\phi_{\alpha,\beta,a}(0)) \phi_{\alpha,\beta,a}(r)} = \frac{(\log a)^{\beta} ||u||_{\mathcal{L}_{p,\omega} \mathcal{B}_{\alpha,a}^{\beta}(\mathbb{B}^n)}}{\omega((\log a)^{\beta}) \phi_{\alpha,\beta,a}(r)},$$

which is what we want. \square

Proof of Theorem 1.3. We divide the proof into two cases according to the values of the parameter p .

Case 5.1. $p \in [1, \infty)$.

Let $y = |y|\xi$ and $\gamma(t) = ty$, where $\xi \in \mathbb{S}^{n-1}$ and $t \in [0, 1]$. Since

$$(5.1) \quad |u(y)| \leq |u(0)| + \int_{\gamma[0,y]} \|Du(\gamma)\| \cdot |d\gamma| = |u(0)| + \int_0^{|y|} \|Du(\rho\xi)\| d\rho,$$

we know from Minkowski's inequality (cf. [32, Theorem 3.5]) that

$$M_p(|y|, u) = \left(\int_{\mathbb{S}^{n-1}} |u(|y|\xi)|^p d\sigma(\xi) \right)^{\frac{1}{p}} \leq |u(0)| + \left(\int_{\mathbb{S}^{n-1}} \left(\int_0^{|y|} \|Du(\rho\xi)\| d\rho \right)^p d\sigma(\xi) \right)^{\frac{1}{p}}.$$

Further, Minkowski's integral inequality (cf. [33, A. 1]) gives

$$M_p(|y|, u) \leq |u(0)| + \int_0^{|y|} \left(\int_{\mathbb{S}^{n-1}} \|Du(\rho\xi)\|^p d\sigma(\xi) \right)^{\frac{1}{p}} d\rho.$$

Since it follows from Lemma 5.1 that

$$\begin{aligned} \int_0^{|y|} \left(\int_{\mathbb{S}^{n-1}} \|Du(\rho\xi)\|^p d\sigma(\xi) \right)^{\frac{1}{p}} d\rho &\leq \frac{(\log a)^\beta \|u\|_{\mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,a}^\beta(\mathbb{B}^n)}}{\omega((\log a)^\beta)} \int_0^{|y|} \frac{1}{\phi_{\alpha,\beta,a}(\rho)} d\rho \\ &= \frac{(\log a)^\beta \|u\|_{\mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,a}^\beta(\mathbb{B}^n)}}{\omega((\log a)^\beta)} \int_0^1 \frac{|y|}{\phi_{\alpha,\beta,a}(\rho|y|)} d\rho, \end{aligned}$$

we know that for all $y \in \mathbb{B}^n$,

$$(5.2) \quad M_p(|y|, u) \leq |u(0)| + \frac{(\log a)^\beta \|u\|_{\mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,a}^\beta(\mathbb{B}^n)}}{\omega((\log a)^\beta)} \int_0^1 \frac{|y|}{\phi_{\alpha,\beta,a}(\rho|y|)} d\rho.$$

Case 5.2. $p = \infty$.

For any $y \in \mathbb{B}^n$, it follows from the assumption $u \in \mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,a}^\beta(\mathbb{B}^n)$ and Lemma 5.1 that

$$\begin{aligned} (5.3) \quad M_\infty(|y|, u) &\leq |u(0)| + \int_0^{|y|} M_\infty(\rho, \|Du\|) d\rho && \text{(by (5.1))} \\ &\leq |u(0)| + \frac{(\log a)^\beta \|u\|_{\mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,a}^\beta(\mathbb{B}^n)}}{\omega((\log a)^\beta)} \int_0^{|y|} \frac{1}{\phi_{\alpha,\beta,a}(\rho)} d\rho \\ &= |u(0)| + \frac{(\log a)^\beta \|u\|_{\mathcal{L}_{p,\omega}\mathcal{B}_{\alpha,a}^\beta(\mathbb{B}^n)}}{\omega((\log a)^\beta)} \int_0^1 \frac{|y|}{\phi_{\alpha,\beta,a}(\rho|y|)} d\rho. \end{aligned}$$

We conclude from (5.2) and (5.3) that the proof of this theorem is complete. \square

Proof of Theorem 1.4. We prove this theorem by considering two possibilities according to the values of the parameter q .

Case 5.3. $q = \infty$.

To prove the theorem in this case, we need to estimate the operator norm: $\|Du(x)\|$. We start with some preparation. First, we shall estimate the quantity $|\nabla u_j(x)|^p$ in terms of the integral $\int_{\mathbb{B}^n(0, \frac{1+4|x|}{4+|x|})} |u_j(y)|^p d\nu(y)$. For this, we take $\delta = \frac{1}{4}$ in Lemma B(2). Then by (2.8), we know that there is a constant μ_{11} such that for any $x \in \mathbb{B}^n$ and each $j \in \{1, \dots, n\}$,

$$(5.4) \quad (1 - |x|^2)^p |\nabla u_j(x)|^p \leq \mu_{11} \int_{E(x, \frac{1}{4})} (1 - |y|^2)^{-n} |u_j(y)|^p d\nu(y),$$

where $\mu_{11} = \mu_{11}(\mu_{08}(p, \frac{1}{4}))$. Moreover, by (2.5), we obtain that

$$E(x, \frac{1}{4}) \subset \mathbb{B}^n \left(0, \frac{1+4|x|}{4+|x|} \right).$$

Then we infer from (2.6) and (5.4) that

$$(5.5) \quad \begin{aligned} |\nabla u_j(x)|^p &\leq \frac{10^n \mu_{11}}{3^n (1 - |x|)^{n+p}} \int_{E(x, \frac{1}{4})} |u_j(y)|^p d\nu(y) \\ &\leq \frac{10^n \mu_{11}}{3^n (1 - |x|)^{n+p}} \int_{\mathbb{B}^n(0, \frac{1+4|x|}{4+|x|})} |u_j(y)|^p d\nu(y). \end{aligned}$$

Second, we shall estimate the integral $\int_{\mathbb{B}^n(0, \frac{1+4|x|}{4+|x|})} |u_j(y)|^p d\nu(y)$. Since the assumption

$$M_p(r, u) = \left(\int_{\mathbb{S}^{n-1}} |u(r\xi)|^p d\sigma(\xi) \right)^{\frac{1}{p}} = O \left(\frac{1}{(1-r)^\alpha} \right)$$

as $r \rightarrow 1^-$, together with the obvious fact $|u(x)| \geq |u_j(x)|$ in \mathbb{B}^n , implies

$$M_p(r, u_j) \leq M_p(r, u),$$

we see that there exists a constant μ_{12} such that

$$(5.6) \quad M_p(r, u_j) \leq \frac{\mu_{12}}{(1-r)^\alpha}$$

as $r \rightarrow 1^-$. It follows from the assumption $\alpha p > 1$ that for each $j \in \{1, \dots, n\}$,

$$(5.7) \quad \begin{aligned} &\int_{\mathbb{B}^n(0, \frac{1+4|x|}{4+|x|})} |u_j(y)|^p d\nu(y) \\ &= \int_0^{\frac{1+4|x|}{4+|x|}} n\rho^{n-1} \left(\int_{\mathbb{S}^{n-1}} |u_j(\rho\xi)|^p d\sigma(\xi) \right) d\rho \\ &\leq \mu_{12}^p \int_0^{\frac{1+4|x|}{4+|x|}} \frac{n\rho^{n-1}}{(1-\rho)^{\alpha p}} d\rho \quad (\text{by (5.6)}) \\ &\leq \frac{n5^{\alpha p-1} \mu_{12}^p}{3^{\alpha p-1} (\alpha p - 1) (1 - |x|)^{\alpha p-1}}. \quad (\text{since } \alpha p > 1) \end{aligned}$$

Now, we are ready to get the estimate on $\|Du(x)\|$. We deduce from (2.1), together with the combination of (5.5) and (5.7), that

$$\|Du(x)\| \leq \left(\sum_{j=1}^n |\nabla u_j(x)|^2 \right)^{\frac{1}{2}} \leq \sqrt{n} \left(\frac{n 10^n 5^{\alpha p-1} \mu_{11} \mu_{12}^p}{3^{n+\alpha p-1} (\alpha p - 1)} \right)^{\frac{1}{p}} \frac{1}{(1-|x|)^{\alpha+1+\frac{n-1}{p}}}.$$

Hence

$$(5.8) \quad M_\infty(r, \|Du\|) = \sup_{\xi \in \mathbb{S}^{n-1}} \|Du(r\xi)\| = O \left(\frac{1}{(1-r)^{\alpha+1+\frac{n-1}{p}}} \right)$$

as $r \rightarrow 1^-$.

Case 5.4. $q \in (0, \infty)$.

Obviously, for all $q \in (0, \infty)$,

$$(5.9) \quad \begin{aligned} M_q(r, \|Du\|) &= \left(\int_{\mathbb{S}^{n-1}} \|Du(r\xi)\|^q d\sigma(\xi) \right)^{\frac{1}{q}} \leq M_\infty(r, \|Du\|) \\ &= O \left(\frac{1}{(1-r)^{\alpha+1+\frac{n-1}{p}}} \right) \end{aligned}$$

as $r \rightarrow 1^-$.

Now, we conclude from (5.8) and (5.9) that for $q \in (0, \infty]$,

$$u \in \mathcal{L}_{q, id} \mathcal{B}_{\alpha+1+\frac{n-1}{p}, a}^0(\mathbb{B}^n),$$

and so the proof of this theorem is finished. \square

6. DIRICHLET ENERGY INTEGRALS OF HYPERBOLIC HARMONIC MAPPINGS

We shall prove Theorem 1.5 in this section. The proof will be based on several lemmas. Before the statements and the proofs of the lemmas, for convenience, we introduce the following notational conventions in \mathbb{B}^n .

Suppose $p \in [2, \infty)$, $r \in [0, 1)$ and $u = (u_1, \dots, u_n) \in C^2(\mathbb{B}^n, \mathbb{R}^n)$. Let

$$(6.1) \quad \Psi_{3,p}(x) = |u(x)|^{p-4} \Psi_1(x), \quad \Psi_{4,p}(x) = |u(x)|^{p-2} \Psi_2(x),$$

$$(6.2) \quad \Psi_{5,p}(x, r) = p(1-r^2)^{n-2} (1-|x|^2)^2 \Psi_{3,p}(x)$$

and

$$\Psi_{6,p}(x, r) = p(1-r^2)^{n-2} (1-|x|^2)^2 \Psi_{4,p}(x),$$

where

$$\Psi_1(x) = \sum_{i=1}^n \left(\sum_{j=1}^n u_j(x) \frac{\partial}{\partial x_i} u_j(x) \right)^2, \quad \Psi_2(x) = \sum_{i=1}^n \left(\sum_{j=1}^n \left(\frac{\partial}{\partial x_i} u_j(x) \right)^2 \right).$$

Here and hereafter, we make the convention: (1) If $p \in [2, 4)$, we allow $|u(x)|^{p-4}$ to take on the value ∞ , that is, $|u(x)|^{p-4} = \infty$ whenever $u(x) = 0$ (cf. [37, Page 27]); (2) $|u(x)|^{p-4} \cdot |u(x)|^2 = |u(x)|^{p-2}$.

Therefore, for $x \in \mathbb{B}^n$, we have

$$\Psi_{3,p}(x) \leq |u(x)|^{p-2} \sum_{j=1}^n |\nabla u_j(x)|^2 < \infty.$$

Further, we have the following inequalities on Ψ_i , which are useful for the discussions in the rest of this section.

$$(6.3) \quad \Psi_1(x) \leq |u(x)|^2 \sum_{j=1}^n |\nabla u_j(x)|^2, \quad \Psi_2(x) = \sum_{j=1}^n |\nabla u_j(x)|^2 \geq \|Du(x)\|^2,$$

$$(6.4) \quad \Psi_{3,p}(x) \leq n|u(x)|^{p-2} \cdot \|Du(x)\|^2$$

and

$$(6.5) \quad |u(x)|^{p-2} \cdot \|Du(x)\|^2 \leq \Psi_{4,p}(x) \leq n|u(x)|^{p-2} \cdot \|Du(x)\|^2.$$

Moreover, since $\left\{(|u(x)|^2 + \frac{1}{n})^{\frac{p}{2}-2}\right\}_{n=1}^{\infty}$ is a collection of measurable mappings in \mathbb{B}^n , by [37, Property 4], we see that

$$|u(x)|^{p-4} = \lim_{n \rightarrow \infty} (|u(x)|^2 + \frac{1}{n})^{\frac{p}{2}-2}$$

is also measurable in \mathbb{B}^n . Thus, for $p \in [2, \infty)$, the mappings $\Psi_1, \Psi_2, \Psi_{3,p}, \Psi_{4,p}$ are all measurable in \mathbb{B}^n , and both $\Psi_{5,p}$ and $\Psi_{6,p}$ are measurable in $\mathbb{B}^n \times [0, 1)$.

Now, we are ready to state and prove our lemmas. The first lemma reads as follows.

Lemma 6.1. *Suppose $\Delta_h u = 0$. Then for $p \in [2, \infty)$,*

$$r^{n-1} \frac{d}{dr} M_p^p(r, u) = \frac{(p-2)}{n} \int_{\mathbb{B}^n(0,r)} \Psi_{5,p}(x, r) d\tau(x) + \frac{1}{n} \int_{\mathbb{B}^n(0,r)} \Psi_{6,p}(x, r) d\tau(x).$$

Proof. We consider the case when $p \in (2, 4)$ and the case when $p \in \{2\} \cup [4, \infty)$, separately.

Case 6.1. $p \in (2, 4)$.

Let

$$F_b(x) = (|u(x)|^2 + b)^{\frac{1}{2}},$$

where $0 < b \leq 1$. Then we know from the assumption $\Delta_h u = 0$ that

$$(6.6) \quad \begin{aligned} \Delta_h(F_b^p(x)) &= p(p-2)(1-|x|^2)^2 (|u(x)|^2 + b)^{\frac{p}{2}-2} \Psi_1(x) \\ &\quad + p(1-|x|^2)^2 (|u(x)|^2 + b)^{\frac{p}{2}-1} \Psi_2(x). \end{aligned}$$

Obviously, for any fixed $r \in (0, 1)$, $\Delta_h(F_b^p)$ is continuous in $\mathbb{B}^n(0, r)$. Moreover, by (6.3) and (6.6), we see that

$$\Delta_h(F_b^p(x)) \leq G_p(x),$$

where

$$G_p(x) = p(1 - |x|^2)^2 \left((p-2) |u(x)|^{p-2} + (|u(x)|^2 + 1)^{\frac{p}{2}-1} \right) \sum_{j=1}^n |\nabla u_j(x)|^2.$$

Clearly, G_p is integrable in $\mathbb{B}^n(0, r)$. We infer from [41, Lemma 3.2(a)] (or [8, Lemma 2.1(a)]) and Lebesgue's Dominated Convergence theorem that

$$\begin{aligned} \lim_{b \rightarrow 0^+} r^{n-1} \frac{d}{dr} \int_{\mathbb{S}^{n-1}} F_b^p(r\xi) d\sigma(\xi) &= \frac{1}{n} (1 - r^2)^{n-2} \lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0, r)} \Delta_h(F_b^p(x)) d\tau(x) \\ &= \frac{1}{n} (1 - r^2)^{n-2} \int_{\mathbb{B}^n(0, r)} \lim_{b \rightarrow 0^+} \Delta_h(F_b^p(x)) d\tau(x), \end{aligned}$$

and so (6.6) gives

$$\begin{aligned} &\lim_{b \rightarrow 0^+} r^{n-1} \frac{d}{dr} \int_{\mathbb{S}^{n-1}} F_b^p(r\xi) d\sigma(\xi) \\ &= \frac{(p-2)}{n} \int_{\mathbb{B}^n(0, r)} \Psi_{5,p}(x, r) d\tau(x) + \frac{1}{n} \int_{\mathbb{B}^n(0, r)} \Psi_{6,p}(x, r) d\tau(x). \end{aligned}$$

Hence, to finish the proof in this case, it remains to check the following claim.

Claim 6.1. For $r \in (0, 1)$,

$$\lim_{b \rightarrow 0^+} r^{n-1} \frac{d}{dr} \int_{\mathbb{S}^{n-1}} (F_b(r\xi))^p d\sigma(\xi) = r^{n-1} \frac{d}{dr} M_p^p(r, u).$$

For $x = r\xi \in \mathbb{B}^n$, define

$$F_0^p(x) = \lim_{b \rightarrow 0^+} F_b^p(x) = \lim_{b \rightarrow 0^+} (|u(x)|^2 + b)^{\frac{p}{2}} = |u(x)|^p.$$

By elementary calculations, we know that for any fixed $r_0 \in (0, 1)$ and $p \in (2, 4)$, the mappings $(r, \xi, b) \mapsto F_b^p(r\xi)$ and

$$(6.7) \quad (r, \xi, b) \mapsto \frac{\partial}{\partial r} (F_b^p(r\xi)) = p(|u(r\xi)|^2 + b)^{\frac{p}{2}-1} \left(\sum_{j=1}^n u_j(r\xi) \langle \nabla u_j(r\xi), \xi \rangle \right)$$

are continuous in $[0, r_0] \times \mathbb{S}^{n-1} \times [0, 1]$. Therefore, for any $r \in [0, r_0]$,

$$\begin{aligned} (6.8) \quad &\lim_{b \rightarrow 0^+} r^{n-1} \frac{d}{dr} \int_{\mathbb{S}^{n-1}} F_b^p(r\xi) d\sigma(\xi) = r^{n-1} \int_{\mathbb{S}^{n-1}} \lim_{b \rightarrow 0^+} \frac{\partial}{\partial r} (F_b^p(r\xi)) d\sigma(\xi) \\ &= pr^{n-1} \int_{\mathbb{S}^{n-1}} |u(r\xi)|^{p-2} \left(\sum_{j=1}^n u_j(r\xi) \langle \nabla u_j(r\xi), \xi \rangle \right) d\sigma(\xi) \quad (\text{by (6.7)}) \\ &= r^{n-1} \int_{\mathbb{S}^{n-1}} \frac{\partial}{\partial r} |u(r\xi)|^p d\sigma(\xi) = r^{n-1} \frac{d}{dr} \int_{\mathbb{S}^{n-1}} |u(r\xi)|^p d\sigma(\xi) \\ &= r^{n-1} \frac{d}{dr} M_p^p(r, u). \end{aligned}$$

By letting $r_0 \rightarrow 1^-$, we see that Claim 6.1 is true.

Case 6.2. $p \in \{2\} \cup [4, \infty)$.

By direct calculations, we have

$$(6.9) \quad \Delta_h(|u(x)|^p) = p(p-2)(1-|x|^2)^2 \Psi_{3,p}(x) + p(1-|x|^2)^2 \Psi_{4,p}(x),$$

which, together with [41, Lemma 3.2(a)] (or [8, Lemma 2.1(a)]), guarantees the following:

$$\begin{aligned} r^{n-1} \frac{d}{dr} M_p^p(r, u) &= \frac{(1-r^2)^{n-2}}{n} \int_{\mathbb{B}^n(0,r)} \Delta_h(|u(x)|^p) d\tau(x) \\ &= \frac{(p-2)}{n} \int_{\mathbb{B}^n(0,r)} \Psi_{5,p}(x, r) d\tau(x) + \frac{1}{n} \int_{\mathbb{B}^n(0,r)} \Psi_{6,p}(x, r) d\tau(x), \end{aligned}$$

as required. \square

The following is our second lemma.

Lemma 6.2. *For $s \in [0, 1)$ and $t \in (0, \infty)$,*

$$\int_s^1 r(1-r)^t(1-r^2)^{n-3} dr \leq \frac{2^{n-3}}{n+t-2} (1-s)^{n+t-2}.$$

Proof. The convergence of the integral $\int_0^1 r(1-r)^t(1-r^2)^{n-3} dr$ is obvious since $t \in (0, \infty)$. Let

$$F(s) = \frac{2^{n-3}}{n+t-2} (1-s)^{n+t-2} - \int_s^1 r(1-r)^t(1-r^2)^{n-3} dr.$$

The facts that

$$F'(s) = (s(1+s)^{n-3} - 2^{n-3})(1-s)^{n+t-3} < 0 \quad \text{and} \quad \lim_{s \rightarrow 1^-} F(s) = 0$$

imply that $F(s) > 0$ in $s \in [0, 1)$, which means that Lemma 6.2 holds. \square

The next lemma plays a key role in the proof of Theorem 1.5.

Lemma 6.3. *Suppose $\Delta_h u = 0$ and*

$$(6.10) \quad \lim_{r \rightarrow 1^-} (1-r)^{\varrho} \int_{\mathbb{S}^{n-1}} |u(r\xi)|^{p-1} \cdot \|Du(r\xi)\| d\sigma(\xi) = 0,$$

where $p \in [2, \infty)$ and $\varrho > 2 + 2^{n-2}n - 2^{n-1} - n$. Then

$$\left(1 - \frac{2^{n-2}(n-2)}{n+\varrho-2}\right) \mathcal{D}_u(\varrho, p-2, 2) \leq \sqrt{n\varrho} \mathcal{D}_u(\varrho-1, p-1, 1).$$

Proof. Without loss of generality, we assume that $\mathcal{D}_u(\varrho-1, p-1, 1) < \infty$. We shall prove this lemma by three steps. First, we get a lower bound for the quantity $\sqrt{n\varrho} \mathcal{D}_u(\varrho-1, p-1, 1)$ in terms of the integral $\int_0^1 (1-r)^{\varrho} \frac{d}{dr} \left(r^{n-1} \left(\frac{d}{dr} M_p^p(r, u) \right) \right) dr$ as shown in (6.12) below.

Since (6.8) implies that

$$\begin{aligned}
\frac{d}{dr}M_p^p(r, u) &= p \int_{\mathbb{S}^{n-1}} |u(r\xi)|^{p-2} \left(\sum_{j=1}^n u_j(r\xi) \langle \nabla u_j(r\xi), \xi \rangle \right) d\sigma(\xi) \\
&\leq p \int_{\mathbb{S}^{n-1}} |u(r\xi)|^{p-1} \left(\sum_{j=1}^n |\nabla u_j(r\xi)|^2 \right)^{\frac{1}{2}} d\sigma(\xi) \quad (\text{by Cauchy's inequality}) \\
&\leq \sqrt{n}p \int_{\mathbb{S}^{n-1}} |u(r\xi)|^{p-1} \cdot \|Du(r\xi)\| d\sigma(\xi), \quad (\text{by (2.1)})
\end{aligned}$$

and thus (6.10) gives

$$\lim_{r \rightarrow 1^-} r^{n-1}(1-r)^\varrho \left(\frac{d}{dr}M_p^p(r, u) \right) = 0.$$

Then

$$\sqrt{n} \int_0^1 r^{n-1}(1-r)^{\varrho-1} \left(\frac{d}{dr}M_p^p(r, u) \right) dr \leq p\mathcal{D}_u(\varrho-1, p-1, 1) < \infty$$

and

$$\begin{aligned}
(6.11) \quad & \varrho \int_0^1 r^{n-1}(1-r)^{\varrho-1} \left(\frac{d}{dr}M_p^p(r, u) \right) dr \\
&= - \int_0^1 r^{n-1} \left(\frac{d}{dr}M_p^p(r, u) \right) d(1-r)^\varrho \\
&= \int_0^1 (1-r)^\varrho \frac{d}{dr} \left(r^{n-1} \left(\frac{d}{dr}M_p^p(r, u) \right) \right) dr. \quad (\text{Integration by parts})
\end{aligned}$$

Hence

$$\begin{aligned}
& \sqrt{n}\varrho p \mathcal{D}_u(\varrho-1, p-1, 1) \\
&= \sqrt{n}\varrho p \int_0^1 nr^{n-1}(1-r)^{\varrho-1} \left(\int_{\mathbb{S}^{n-1}} |u(r\xi)|^{p-1} \cdot \|Du(r\xi)\| d\sigma(\xi) \right) dr \\
&\geq n\varrho \int_0^1 r^{n-1}(1-r)^{\varrho-1} \left(\frac{d}{dr}M_p^p(r, u) \right) dr \\
&= n \int_0^1 (1-r)^\varrho \frac{d}{dr} \left(r^{n-1} \left(\frac{d}{dr}M_p^p(r, u) \right) \right) dr, \quad (\text{by (6.11)})
\end{aligned}$$

which leads to

$$(6.12) \quad \sqrt{n}\varrho p \mathcal{D}_u(\varrho-1, p-1, 1) \geq n \int_0^1 (1-r)^\varrho \frac{d}{dr} \left(r^{n-1} \left(\frac{d}{dr}M_p^p(r, u) \right) \right) dr.$$

Second, we shall show that the lower bound of $\sqrt{n}\varrho p\mathcal{D}_u(\varrho-1, p-1, 1)$ in (6.12) can be bounded by a new one related to the functions $\Psi_{3,p}$ and $\Psi_{4,p}$ as formulated in (6.20) below. We shall exploit Lemma 6.1 to reach this goal.

For $r \in [0, 1)$, (6.4) implies that

$$\int_{\mathbb{B}^n(0,r)} (1 - |x|^2)^{2-n} \Psi_{3,p}(x) d\nu(x) < \infty.$$

Therefore, by (6.2) and calculations, we see that

$$(6.13) \quad \frac{d}{dr} \left(\int_{\mathbb{B}^n(0,r)} \Psi_{5,p}(x, r) d\tau(x) \right) = np \int_{\mathbb{S}^{n-1}} r^{n-1} \Psi_{3,p}(r\xi) d\sigma(\xi) \\ - 2p(n-2)r(1-r^2)^{n-3} \int_{\mathbb{B}^n(0,r)} (1 - |x|^2)^{2-n} \Psi_{3,p}(x) d\nu(x)$$

for $r \in [0, 1)$ a.e.. Similarly, we get

$$(6.14) \quad \frac{d}{dr} \left(\int_{\mathbb{B}^n(0,r)} \Psi_{6,p}(x, r) d\tau(x) \right) = np \int_{\mathbb{S}^{n-1}} r^{n-1} \Psi_{4,p}(r\xi) d\sigma(\xi) \\ - 2p(n-2)r(1-r^2)^{n-3} \int_{\mathbb{B}^n(0,r)} (1 - |x|^2)^{2-n} \Psi_{4,p}(x) d\nu(x)$$

for $r \in [0, 1)$ a.e.. Then we deduce from Lemma 6.1, (6.13) and (6.14) that for $r \in (0, 1)$ a.e.,

$$(6.15) \quad \frac{d}{dr} \left(r^{n-1} \frac{d}{dr} M_p^p(r, u) \right) \\ = \frac{(p-2)}{n} \frac{d}{dr} \left(\int_{\mathbb{B}^n(0,r)} \Psi_{5,p}(x, r) d\tau(x) \right) + \frac{1}{n} \frac{d}{dr} \left(\int_{\mathbb{B}^n(0,r)} \Psi_{6,p}(x, r) d\tau(x) \right) \\ = p(p-2) \int_{\mathbb{S}^{n-1}} r^{n-1} \Psi_{3,p}(r\xi) d\sigma(\xi) + p \int_{\mathbb{S}^{n-1}} r^{n-1} \Psi_{4,p}(r\xi) d\sigma(\xi) \\ - \frac{2(n-2)p(p-2)}{n} r(1-r^2)^{n-3} \int_{\mathbb{B}^n(0,r)} (1 - |x|^2)^{2-n} \Psi_{3,p}(x) d\nu(x) \\ - \frac{2(n-2)p}{n} r(1-r^2)^{n-3} \int_{\mathbb{B}^n(0,r)} (1 - |x|^2)^{2-n} \Psi_{4,p}(x) d\nu(x).$$

By substituting (6.15) into (6.12), we have

$$(6.16) \quad \sqrt{n} \varrho p \mathcal{D}_u(\varrho - 1, p - 1, 1) \\ \geq p(p-2) \int_{\mathbb{B}^n} (1 - |x|)^\varrho \Psi_{3,p}(x) d\nu(x) + p \int_{\mathbb{B}^n} (1 - |x|)^\varrho \Psi_{4,p}(x) d\nu(x) \\ - 2(n-2)(p-2) \int_0^1 \left(\int_{\mathbb{B}^n(0,r)} \frac{r(1-r)^\varrho \Psi_{5,p}(x, r)}{(1-r^2)(1-|x|^2)^n} d\nu(x) \right) dr \\ - 2(n-2) \int_0^1 \left(\int_{\mathbb{B}^n(0,r)} \frac{r(1-r)^\varrho \Psi_{6,p}(x, r)}{(1-r^2)(1-|x|^2)^n} d\nu(x) \right) dr,$$

where the equality

$$\begin{aligned} & np \int_0^1 (1-r)^\varrho \left(\int_{\mathbb{S}^{n-1}} r^{n-1} ((p-2)\Psi_{3,p}(r\xi) + \Psi_{4,p}(r\xi)) d\sigma(\xi) \right) dr \\ &= p(p-2) \int_{\mathbb{B}^n} (1-|x|)^\varrho \Psi_{3,p}(x) d\nu(x) + p \int_{\mathbb{B}^n} (1-|x|)^\varrho \Psi_{4,p}(x) d\nu(x) \end{aligned}$$

is applied.

For $i \in \{5, 6\}$, we let

$$(6.17) \quad F_{i,p}(x, r) = \frac{r(1-r)^\varrho}{(1-r^2)(1-|x|^2)^n} \Psi_{i,p}(x, r).$$

For $p \in [2, \infty)$, it follows from the fact “ $\Psi_{i,p}$ ($i \in \{5, 6\}$) being measurable in $\mathbb{B}^n \times [0, 1]$ ” that both $F_{i,p}(x, r)$ are non-negative and measurable in $\mathbb{B}^n \times [0, 1]$. By Fubini’s theorem (cf. [32, p. 165]) and by letting $x = \rho\xi$ ($\xi \in \mathbb{S}^{n-1}$), we get

$$\begin{aligned} \int_0^1 \left(\int_{\mathbb{B}^n(0,r)} F_{5,p}(x, r) d\nu(x) \right) dr &= n \int_{\mathbb{S}^{n-1}} \int_0^1 \int_0^r \rho^{n-1} F_{5,p}(\rho\xi, r) d\rho dr d\sigma(\xi) \\ &= n \int_{\mathbb{S}^{n-1}} \int_0^1 \int_\rho^1 \rho^{n-1} F_{5,p}(\rho\xi, r) dr d\rho d\sigma(\xi), \end{aligned}$$

and so (6.1), (6.2) and (6.17) lead to

$$\begin{aligned} (6.18) \quad & \int_0^1 \left(\int_{\mathbb{B}^n(0,r)} F_{5,p}(x, r) d\nu(x) \right) dr = \int_{\mathbb{B}^n} \left(\int_{|x|}^1 F_{5,p}(x, r) dr \right) d\nu(x) \\ &= p \int_{\mathbb{B}^n} \left(\int_{|x|}^1 r(1-r)^\varrho (1-r^2)^{n-3} dr \right) (1-|x|^2)^{2-n} \Psi_{3,p}(x) d\nu(x). \end{aligned}$$

Similarly, we can have

$$\begin{aligned} (6.19) \quad & \int_0^1 \left(\int_{\mathbb{B}^n(0,r)} F_{6,p}(x, r) d\nu(x) \right) dr = \int_{\mathbb{B}^n} \left(\int_{|x|}^1 F_{6,p}(x, r) dr \right) d\nu(x) \\ &= p \int_{\mathbb{B}^n} \left(\int_{|x|}^1 r(1-r)^\varrho (1-r^2)^{n-3} dr \right) (1-|x|^2)^{2-n} \Psi_{4,p}(x) d\nu(x). \end{aligned}$$

The combination of (6.16) \sim (6.19) guarantees the following:

$$\begin{aligned} (6.20) \quad \sqrt{n} \varrho p \mathcal{D}_u(\varrho-1, p-1, 1) &\geq p(p-2) \int_{\mathbb{B}^n} (1-|x|)^\varrho \Psi_{3,p}(x) d\nu(x) \\ &\quad + p \int_{\mathbb{B}^n} (1-|x|)^\varrho \Psi_{4,p}(x) d\nu(x) \\ &\quad - (p-2) \Phi_{3,p}(x) - \Phi_{4,p}(x), \end{aligned}$$

where for $i \in \{3, 4\}$,

$$\Phi_{i,p}(x) = 2(n-2)p \int_{\mathbb{B}^n} \left(\int_{|x|}^1 r(1-r)^\varrho (1-r^2)^{n-3} dr \right) (1-|x|^2)^{2-n} \Psi_{i,p}(x) d\nu(x).$$

Third, we shall finish the proof of this lemma by applying Lemma 6.2.

Since $\varrho > 2 + 2^{n-2}n - 2^{n-1} - n \geq 0$, we take $t = \varrho$ and $s = |x|$ in Lemma 6.2. Then for each $i \in \{3, 4\}$,

$$\Phi_{i,p}(x) \leq \frac{2^{n-2}p(n-2)}{n + \varrho - 2} \int_{\mathbb{B}^n} (1 + |x|)^{2-n} (1 - |x|)^{\varrho} \Psi_{i,p}(x) d\nu(x),$$

and so, it follows from (6.20) that

$$\begin{aligned} & \sqrt{n}\varrho \mathcal{D}_u(\varrho - 1, p - 1, 1) \\ & \geq (p - 2) \int_{\mathbb{B}^n} (1 - |x|)^{\varrho} \Psi_{3,p}(x) d\nu(x) + \int_{\mathbb{B}^n} (1 - |x|)^{\varrho} \Psi_{4,p}(x) d\nu(x) \\ & \quad - \frac{2^{n-2}(n-2)(p-2)}{n + \varrho - 2} \int_{\mathbb{B}^n} (1 + |x|)^{2-n} (1 - |x|)^{\varrho} \Psi_{3,p}(x) d\nu(x) \\ & \quad - \frac{2^{n-2}(n-2)}{\varrho + n - 2} \int_{\mathbb{B}^n} (1 + |x|)^{2-n} (1 - |x|)^{\varrho} \Psi_{4,p}(x) d\nu(x) \\ & \geq \left(1 - \frac{2^{n-2}(n-2)}{n + \varrho - 2}\right) \int_{\mathbb{B}^n} (1 - |x|)^{\varrho} \Psi_{4,p}(x) d\nu(x). \quad (\text{since } \varrho > 2 + 2^{n-2}n - 2^{n-1} - n). \end{aligned}$$

Thus, (6.5) shows that

$$\sqrt{n}\varrho \mathcal{D}_u(\varrho - 1, p - 1, 1) \geq \left(1 - \frac{2^{n-2}(n-2)}{n + \varrho - 2}\right) \mathcal{D}_u(\varrho, p - 2, 2),$$

which is what we need. \square

Proof of Theorem 1.5. To prove this theorem, first, we find an upper bound for $|u(x)|^{2\beta-2}$ involving the factor $(1 - |x|)^{(\beta-1)(n+\beta-2)}$ as stated in the following claim.

Claim 6.2. *There exists a positive constant μ_{13} such that*

$$|u(x)|^{2\beta-2} \leq \frac{\mu_{13}}{(1 - |x|)^{(\beta-1)(n+\beta-2)}},$$

where $\mu_{13} = \mu_{13}(n, \beta, \mu_{08}(2, \frac{1}{4}), |u(0)|, \mathcal{D}_u(\beta - 1, 1, 1))$ is defined in (6.24) below.

We start with an estimate on $|u(x)|$. Since

$$(6.21) \quad |u(x)| \leq |u(0)| + \int_{\gamma_{[0,x]}} \|Du(y)\| \cdot |dy|,$$

we easily see that an upper bound for $\|Du(x)\|$ is needed. For this, we need some preparation.

Take $\delta = \frac{1}{4}$ and $p = 2$ in Lemma B(1). Then we know from (2.8) that there is a constant $\mu_{14} = \mu_{14}(\mu_{08}(2, \frac{1}{4}))$ such that for each $j \in \{1, \dots, n\}$,

$$(1 - |x|^2)^2 |\nabla u_j(x)|^2 \leq \mu_{14} \int_{E(x, \frac{1}{4})} (1 - |y|^2)^{2-n} |\nabla u_j(y)|^2 d\nu(y).$$

Since (2.6) implies

$$(1 - |y|^2)^{2-n} \leq \frac{10^{n+\beta-2}(1 - |y|^2)^{\beta}}{3^{n+\beta-2}(1 - |x|^2)^{n+\beta-2}},$$

further, we get that

$$(1 - |x|^2)^2 |\nabla u_j(x)|^2 \leq \frac{\mu_{14} 10^{n+\beta-2}}{(3(1 - |x|^2))^{n+\beta-2}} \int_{E(x, \frac{1}{4})} (1 - |y|^2)^\beta |\nabla u_j(y)|^2 d\nu(y),$$

and so,

$$(6.22) \quad (1 - |x|^2)^{n+\beta} |\nabla u_j(x)|^2 \leq \mu_{14} \frac{10^{n+\beta-2}}{3^{n+\beta-2}} \int_{E(x, \frac{1}{4})} (1 - |y|^2)^\beta |\nabla u_j(y)|^2 d\nu(y).$$

Now, we are ready to find an upper bound for $\|Du(x)\|$ (see (6.28) below). We infer that

$$\begin{aligned} & (1 - |x|^2)^{n+\beta} \|Du(x)\|^2 \\ & \leq (1 - |x|^2)^{n+\beta} \sum_{j=1}^n |\nabla u_j(x)|^2 \quad (\text{by (2.1)}) \\ & \leq \mu_{14} \frac{10^{n+\beta-2}}{3^{n+\beta-2}} \int_{E(x, \frac{1}{4})} (1 - |y|^2)^\beta \left(\sum_{j=1}^n |\nabla u_j(y)|^2 \right) d\nu(y) \quad (\text{by (6.22)}) \\ & \leq n \mu_{14} \frac{10^{n+\beta-2}}{3^{n+\beta-2}} \int_{\mathbb{B}^n(0,1)} (1 - |y|^2)^\beta \|Du(x)\|^2 d\nu(y). \quad (\text{by (2.1)}) \end{aligned}$$

By taking $p = 2$ and $\varrho = \beta$, we know from Lemma 6.3, together with the assumption $\beta > 2 + 2^{n-2}n - 2^{n-1} - n$, that

$$(6.23) \quad (1 - |x|^2)^{n+\beta} \|Du(x)\|^2 \leq \frac{\mu_{14} \beta (n + \beta - 2) n^{3/2} 10^{n+\beta-2}}{(n + \beta - 2 - 2^{n-2}n + 2^{n-1}) 3^{n+\beta-2}} \mathcal{D}_u(\beta - 1, 1, 1).$$

Without loss of generality, we may assume that $\mathcal{D}_u(\beta - 1, 1, 1) > 0$.

For convenience, we let

$$(6.24) \quad \mu_{13} = \begin{cases} 4^{\beta-1} (|u(0)|^{2\beta-2} + (2\mu_{15})^{2\beta-2} (n + \beta - 2)^{2-2\beta}), & \text{if } u(0) \neq 0, \\ (2\mu_{15})^{2\beta-2} (n + \beta - 2)^{2-2\beta}, & \text{if } u(0) = 0. \end{cases}$$

where

$$(6.25) \quad \mu_{15} = \sqrt{\frac{\mu_{14} \beta (n + \beta - 2) n^{3/2} 10^{n+\beta-2}}{(n + \beta - 2 - 2^{n-2}n + 2^{n-1}) 3^{n+\beta-2}} \mathcal{D}_u(\beta - 1, 1, 1)}.$$

Obviously, μ_{13} depends only on n , β , $\mu_{08}(2, \frac{1}{4})$, $|u(0)|$ and $\mathcal{D}_u(\beta - 1, 1, 1)$, and

$$(6.26) \quad \mu_{13} \leq 2^{2\beta-1} \max \left\{ |u(0)|^{2\beta-2}, \mu_{16} (\mathcal{D}_u(\beta - 1, 1, 1))^{\beta-1} \right\},$$

where

$$(6.27) \quad \mu_{16} = \left(\frac{4\mu_{14} \beta n^{3/2} 10^{n+\beta-2}}{3^{n+\beta-2} (n + \beta - 2) (n + \beta - 2 - 2^{n-2}n + 2^{n-1})} \right)^{\beta-1}.$$

Hence it easily follows from (6.23) and (6.25) that

$$(6.28) \quad \|Du(x)\| \leq \frac{\mu_{15}}{(1 - |x|)^{\frac{n+\beta}{2}}},$$

which is what we need.

By using (6.28), together with (6.21), we get the following upper bound for $|u(x)|$:

$$(6.29) \quad |u(x)| \leq |u(0)| + \mu_{15} \int_{\gamma_{[0,x]}} \frac{|dy|}{(1-|y|)^{\frac{n+\beta}{2}}} \leq |u(0)| + \frac{2\mu_{15}}{(n+\beta-2)(1-|x|)^{\frac{n+\beta}{2}-1}}.$$

It is the time for us to finish the proof of the claim. If $u(0) \neq 0$, then the upper bound (6.29) on $|u(x)|$ along with the assumption $\beta \geq 1$ leads to

$$\begin{aligned} |u(x)|^{2\beta-2} &\leq 4^{\beta-1} \left(|u(0)|^{2\beta-2} + \frac{(2\mu_{15})^{2\beta-2}(n+\beta-2)^{2-2\beta}}{(1-|x|)^{(\beta-1)(n+\beta-2)}} \right) \\ &\leq 4^{\beta-1} \frac{|u(0)|^{2\beta-2} + (2\mu_{15})^{2\beta-2}(n+\beta-2)^{2-2\beta}}{(1-|x|)^{(\beta-1)(n+\beta-2)}} \quad (\text{since } \beta \geq 1) \\ &= \frac{\mu_{13}}{(1-|x|)^{(\beta-1)(n+\beta-2)}}. \quad (\text{by (6.24)}) \end{aligned}$$

Here the inequality (cf. [19, Inequality (4.2)])

$$(a+b)^q \leq 2^q(a^q + b^q)$$

for $a, b \in (0, \infty)$ and $q \in [0, \infty)$ is applied.

If $u(0) = 0$, then (6.24) and (6.29) imply that

$$|u(x)|^{2\beta-2} \leq \frac{(2\mu_{15})^{2\beta-2}(n+\beta-2)^{2-2\beta}}{(1-|x|)^{(\beta-1)(n+\beta-2)}} = \frac{\mu_{13}}{(1-|x|)^{(\beta-1)(n+\beta-2)}},$$

as required.

In the following, we shall finish the proof of the theorem based on Claim 6.2. We separate the discussions into two cases.

Case 6.3. $\beta \in \{1\} \cup [2, \infty)$ and satisfies the related assumption in Theorem 1.5.

Let $\kappa = 2\beta$. Then $\kappa \in \{2\} \cup [4, \infty)$, and so it follows that

$$\begin{aligned} \Delta_h(|u(x)|^\kappa) &= \kappa(\kappa-2)(1-|x|^2)^2 \Psi_{3,\kappa}(x) + \kappa(1-|x|^2)^2 \Psi_{4,\kappa}(x) \quad (\text{by (6.9)}) \\ &\leq n\kappa(\kappa-1)(1-|x|^2)^2 |u(x)|^{\kappa-2} \|Du(x)\|^2 \quad (\text{by (6.4) and (6.5)}) \\ &\leq 2n\beta(2\beta-1)\mu_{13} \frac{(1-|x|^2)^2 \|Du(x)\|^2}{(1-|x|)^{(\beta-1)(n+\beta-2)}}. \quad (\text{by Claim 6.2}) \end{aligned}$$

Thus

$$(1-|x|)^{\beta n + \beta^2 - 2\beta - n} \Delta_h(|u(x)|^{2\beta}) \leq 8n\beta(2\beta-1)\mu_{13}(1-|x|)^\beta \|Du(x)\|^2.$$

By integration, we have

$$\int_{\mathbb{B}^n} (1-|x|)^{\beta n + \beta^2 - 2\beta - n} \Delta_h(|u(x)|^{2\beta}) d\nu(x) \leq 8n\beta(2\beta-1)\mu_{13} \mathcal{D}_u(\beta, 0, 2),$$

and so

$$\begin{aligned}
(6.30) \quad & \int_{\mathbb{B}^n} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} \Delta_h(|u(x)|^{2\beta}) d\nu(x) \\
& \leq \frac{8n^{3/2}\beta^2(2\beta - 1)(n + \beta - 2)\mu_{13}}{n + \beta - 2 - 2^{n-2}n + 2^{n-1}} \mathcal{D}_u(\beta - 1, 1, 1) \quad (\text{by Lemma 6.3}) \\
& \leq \mu_{17} \max \left\{ |u(0)|^{2\beta-2} \mathcal{D}_u(\beta - 1, 1, 1), \mu_{16} (\mathcal{D}_u(\beta - 1, 1, 1))^\beta \right\}, \quad (\text{by (6.26)})
\end{aligned}$$

where $\mu_{17} = \frac{4^{\beta+1}n^{3/2}\beta^2(2\beta-1)(n+\beta-2)}{n+\beta-2-2^{n-2}n+2^{n-1}}$ and μ_{16} is defined in (6.27).

Case 6.4. $\beta \in (1, 2)$ and satisfies the related assumption in Theorem 1.5.

In this case, we let

$$\kappa = 2\beta \quad \text{and} \quad F_b(x) = (|u(x)|^2 + b)^{\frac{1}{2}},$$

where $0 < b \leq 1$.

The similar reasoning as in the discussions in Case 6.1 in Lemma 6.1 guarantees that

$$\begin{aligned}
(6.31) \quad & \lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0, r)} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} \Delta_h(F_b^\kappa(x)) d\nu(x) \\
& = \int_{\mathbb{B}^n(0, r)} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} \lim_{b \rightarrow 0^+} (\Delta_h(F_b^\kappa(x))) d\nu(x) \\
& = \kappa \int_{\mathbb{B}^n(0, r)} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} (1 - |x|^2)^2 ((\kappa - 2)\Psi_{3, \kappa}(x) + \Psi_{4, \kappa}(x)) d\nu(x),
\end{aligned}$$

where $\Psi_{3, \kappa}$ and $\Psi_{4, \kappa}$ are defined in (6.1) by taking $p = \kappa$ in (6.1). Let $C_0^\infty(\mathbb{B}^n(0, r), \mathbb{R})$ denote the class of compactly supported functions with derivatives of all orders (cf. [17, Definition A.13]).

Similarly, for all $\varphi \in C_0^\infty(\mathbb{B}^n(0, r), \mathbb{R})$, by Lebesgue's Dominated Convergence theorem, we have

$$\begin{aligned}
(6.32) \quad & \lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0, r)} \varphi(x) \Delta(F_b^\kappa(x)) d\nu(x) \\
& = \int_{\mathbb{B}^n(0, r)} \varphi(x) \lim_{b \rightarrow 0^+} (\Delta(F_b^\kappa(x))) d\nu(x) \\
& = \int_{\mathbb{B}^n(0, r)} \varphi(x) Q_\kappa(x) d\nu(x),
\end{aligned}$$

where

$$\begin{aligned}
(6.33) \quad \Delta(F_b^\kappa(x)) & = \kappa(\kappa - 2)(|u(x)|^2 + b)^{\frac{\kappa}{2} - 2} \Psi_1(x) \\
& \quad + \kappa(|u(x)|^2 + b)^{\frac{\kappa}{2} - 1} \left(\Psi_2(x) + \sum_{j=1}^n u_j \Delta u_j \right)
\end{aligned}$$

and

$$(6.34) \quad Q_\kappa(x) = \kappa(\kappa - 2)|u(x)|^{\kappa-4}\Psi_1(x) + \kappa|u(x)|^{\kappa-2} \left(\Psi_2(x) + \sum_{j=1}^n u_j \Delta u_j \right) \in L^1(\mathbb{B}^n(0, r), \mathbb{R}).$$

Further, (6.4), (6.5) and Claim 6.2 lead to

$$(6.35) \quad \begin{aligned} & (1 - |x|^2)^2((\kappa - 2)\Psi_{3,\kappa}(x) + \Psi_{4,\kappa}(x)) \\ & \leq 4n(\kappa - 1)(1 - |x|)^2|u(x)|^{\kappa-2}\|Du(x)\|^2 \\ & \leq \frac{4n(2\beta - 1)\mu_{13}\|Du(x)\|^2}{(1 - |x|)^{\beta n + \beta^2 - 3\beta - n}}, \quad (\text{since } \kappa = 2\beta) \end{aligned}$$

which implies

$$(6.36) \quad \begin{aligned} & \lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0, r)} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} \Delta_h(F_b^{2\beta}(x)) \, d\nu(x) \\ & \leq 8n\beta(2\beta - 1)\mu_{13} \int_{\mathbb{B}^n(0, r)} (1 - |x|)^\beta \|Du(x)\|^2 \, d\nu(x) \quad (\text{by (6.31) and (6.35)}) \\ & = 8n\beta(2\beta - 1)\mu_{13}\mathcal{D}_u(\beta, 0, 2) \\ & \leq \frac{8n^{3/2}\beta^2(2\beta - 1)(n + \beta - 2)\mu_{13}}{n + \beta - 2 - 2^{n-2}n + 2^{n-1}} \mathcal{D}_u(\beta - 1, 1, 1) \quad (\text{by Lemma 6.3}) \\ & \leq \mu_{17} \max \left\{ |u(0)|^{2\beta-2} \mathcal{D}_u(\beta - 1, 1, 1), \mu_{16}(\mathcal{D}_u(\beta - 1, 1, 1))^\beta \right\}, \quad (\text{by (6.26)}) \end{aligned}$$

where μ_{16} is defined in (6.27).

Next, we will prove the following inequality holds:

$$(6.37) \quad \begin{aligned} & \int_{\mathbb{B}^n(0, r)} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} \Delta_h(|u(x)|^{2\beta}) \, d\nu(x) \\ & \leq \mu_{17} \max \left\{ |u(0)|^{2\beta-2} \mathcal{D}_u(\beta - 1, 1, 1), \mu_{16}(\mathcal{D}_u(\beta - 1, 1, 1))^\beta \right\}. \end{aligned}$$

In order to prove (6.37), we need a claim.

Claim 6.3. For $r \in (0, 1)$,

$$\begin{aligned} & \lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0, r)} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} \Delta_h(F_b^{2\beta}(x)) \, d\nu(x) \\ & = \int_{\mathbb{B}^n(0, r)} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} \Delta_h(|u(x)|^{2\beta}) \, d\nu(x). \end{aligned}$$

For any fixed $r \in (0, 1)$, since the mappings

$$(x, b) \mapsto F_b^{2\beta}(x) \quad \text{and} \quad (x, b) \mapsto \frac{\partial}{\partial x_i} F_b^{2\beta}(x) = 2\beta(|u(x)|^2 + b)^{\beta-1} \left(\sum_{j=1}^n u_j(x) \frac{\partial}{\partial x_i} u_j(x) \right)$$

are continuous in $\overline{\mathbb{B}^n}(0, r) \times [0, 1]$, we see that

$$\begin{aligned}
(6.38) \quad & \int_{\mathbb{B}^n(0, r)} \varphi(x) \Delta(|u(x)|^{2\beta}) \, d\nu(x) \\
&= - \int_{\mathbb{B}^n(0, r)} \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} |u(x)|^{2\beta} \cdot \frac{\partial}{\partial x_i} \varphi(x) \right) \, d\nu(x) \quad (\text{cf. [17, Definition A.13]}) \\
&= - \lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0, r)} \sum_{i=1}^n \left(\frac{\partial}{\partial x_i} F_b^{2\beta}(x) \cdot \frac{\partial}{\partial x_i} \varphi(x) \right) \, d\nu(x) \\
&= \lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0, r)} \varphi(x) \Delta(F_b^{2\beta}(x)) \, d\nu(x) \quad (\text{by Stokes' theorem})
\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{B}^n(0, r), \mathbb{R})$. Then

$$\begin{aligned}
\int_{\mathbb{B}^n(0, r)} \varphi(x) \Delta(|u(x)|^{2\beta}) \, d\nu(x) &= \lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0, r)} \varphi(x) \Delta(F_b^{2\beta}(x)) \, d\nu(x) \quad (\text{by (6.38)}) \\
&= \int_{\mathbb{B}^n(0, r)} \varphi(x) Q_{2\beta}(x) \, d\nu(x). \quad (\text{by (6.32)})
\end{aligned}$$

Therefore,

$$(6.39) \quad \Delta(|u(x)|^{2\beta}) = Q_{2\beta}(x) \in L^1(\mathbb{B}^n(0, r), \mathbb{R})$$

in $\mathbb{B}^n(0, r)$ a.e., where $Q_{2\beta}$ is defined in (6.34) by taking $\kappa = 2\beta$ in (6.34). Further, (6.3) and (6.33) guarantee that

$$|\Delta F_b^{2\beta}(x)| \leq R_{2\beta}(x),$$

where

$$\begin{aligned}
R_{2\beta}(x) &= 4n\beta(\beta - 1)|u(x)|^{2\beta-2} \|Du(x)\|^2 \\
&\quad + 2\beta(|u(x)|^2 + 1)^{\beta-1} (n\|Du(x)\|^2 + |u(x)|^2 + |\Delta u|^2).
\end{aligned}$$

For $b \in (0, 1]$, clearly, $\{|\Delta F_b^{2\beta}|\}$ and $R_{2\beta}$ are integrable in $\mathbb{B}^n(0, r)$, and

$$\lim_{b \rightarrow 0^+} \left| \Delta F_b^{2\beta}(x) \right| = |Q_{2\beta}(x)|.$$

Then Lebesgue's Dominated Convergence theorem implies that

$$\lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0, r)} \left| \Delta F_b^{2\beta}(x) \right| \, d\nu(x) = \int_{\mathbb{B}^n(0, r)} |Q_{2\beta}(x)| \, d\nu(x).$$

Thus, by [30, Problem 4.33], we see

$$\lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0, r)} \left| \Delta F_b^{2\beta}(x) - Q_{2\beta}(x) \right| \, d\nu(x) = 0,$$

which shows that

$$\begin{aligned}
& \lim_{b \rightarrow 0^+} \int_{\mathbb{B}^n(0,r)} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} (1 - |x|^2)^2 \Delta(F_b^{2\beta}(x)) \, d\nu(x) \\
&= \int_{\mathbb{B}^n(0,r)} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} (1 - |x|^2)^2 Q_{2\beta}(x) \, d\nu(x) \\
&= \int_{\mathbb{B}^n(0,r)} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} (1 - |x|^2)^2 \Delta(|u(x)|^{2\beta}) \, d\nu(x). \quad (\text{by (6.39)})
\end{aligned}$$

Then the fact that the mappings

$$(x, b) \mapsto F_b^{2\beta}(x) \quad \text{and} \quad (x, b) \mapsto \frac{\partial}{\partial x_i} F_b^{2\beta}(x)$$

are continuous in $\overline{\mathbb{B}^n}(0, r) \times [0, 1]$ implies that Claim 6.3 holds, and hence we can easily conclude (6.37) from (6.36) and Claim 6.3.

Let $r \rightarrow 1^-$ in (6.37) and take $\mu_3 = \mu_{17}(1 + \mu_{16})$. Then under the assumptions in this case, it follows from (6.30) and (6.37) that

$$\begin{aligned}
& \int_{\mathbb{B}^n} (1 - |x|)^{\beta n + \beta^2 - 2\beta - n} \Delta_h(|u(x)|^{2\beta}) \, d\nu(x) \\
& \leq \mu_3 \max \left\{ |u(0)|^{2\beta-2} \mathcal{D}_u(\beta - 1, 1, 1), (\mathcal{D}_u(\beta - 1, 1, 1))^\beta \right\}.
\end{aligned}$$

So the proof of Theorem 1.5 is complete. \square

7. WEAK UNIFORM BOUNDEDNESS AND QUASIHYPERSBOLIC METRIC

In this section, we shall prove Theorem 1.6.

Proof of Theorem 1.6. The sufficiency in the theorem easily follows from the proof of [23, Theorem 2.8] or the proof of [6, Theorem 2]. So we only need to prove the necessity. Obviously, for every $x \in \mathbb{B}^n$ and $y \in \overline{\mathbb{B}^n}\left(x, \frac{d_{\mathbb{B}^n}(x)}{4}\right)$,

$$d_{\mathbb{B}^n}(y) \geq d_{\mathbb{B}^n}(x) - \frac{1}{4}d_{\mathbb{B}^n}(x) = \frac{3}{4}d_{\mathbb{B}^n}(x).$$

Hence

$$|x - y| \leq \frac{d_{\mathbb{B}^n}(x)}{4} \leq \min \left\{ \frac{d_{\mathbb{B}^n}(x)}{2}, \frac{d_{\mathbb{B}^n}(y)}{2} \right\},$$

and so

$$r_{\mathbb{B}^n}(x, y) \leq \frac{1}{2}.$$

Then the assumption “ u satisfying the weak uniform boundedness property” implies that

$$|u(y) - u(x)| \leq \mu_{04} d_{u(\mathbb{B}^n)}(u(x)),$$

where μ_{04} is the constant from (1.4). From Lemma 3.1, we deduce that

$$(7.1) \quad ||Du(x)|| \leq \sqrt{n} \mu_{04} \mu_{09} |\mathbb{B}^n| \frac{5^n \cdot d_{u(\mathbb{B}^n)}(u(x))}{8^n \cdot d_{\mathbb{B}^n}(x)}.$$

Since $|du(z)| \leq ||Du(z)|| \cdot |dz|$, by taking

$$\mu_4 = \frac{5^n}{8^n} \sqrt{n} \mu_{04} \mu_{09} |\mathbb{B}^n|,$$

it follows from (7.1) that for all $x, y \in \mathbb{B}^n$,

$$\begin{aligned} k_{u(\mathbb{B}^n)}(u(x), u(y)) &\leq \inf_{\gamma \in \Gamma_{xy}(\mathbb{B}^n)} \int_{\gamma} \frac{1}{d_{u(\mathbb{B}^n)}(u(z))} |du(z)| \\ &\leq \inf_{\gamma \in \Gamma_{xy}(\mathbb{B}^n)} \int_{\gamma} \frac{||Du(z)||}{d_{u(\mathbb{B}^n)}(u(z))} |dz| \\ &\leq \inf_{\gamma \in \Gamma_{xy}(\mathbb{B}^n)} \int_{\gamma} \frac{\mu_4}{d_{\mathbb{B}^n}(z)} |dz| \\ &= \mu_4 k_{\mathbb{B}^n}(x, y), \end{aligned}$$

and so Theorem 1.6 is proved. \square

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